Degenerate Elliptic Partial Differential Equations in Rough Geometries

By Nicholas Bigley, B.Eng.

A thesis submitted in partial fulfillment of the Honours Thesis Course, MATH-4900.

Cape Breton University
May 2025

Abstract

In this thesis, we investigate the existence of weak solutions to the degenerate, linear, elliptic second-order partial differential equation

$$\begin{cases} -\frac{1}{w}\mathrm{Div}\,(Q\nabla u) + \mathbf{H}\mathbf{T}u + Fu &= f, \quad x \in \Omega; \\ u &= 0, \quad x \in \partial\Omega; \end{cases} \tag{0.1}$$

on an open, connected, bounded set $\Omega \subset \mathbb{R}^n$. In this equation, Q is a non-negative definite, $n \times n$ symmetric matrix that may be singular at a finite number of points and $w \in L^1_{loc}(\Omega)$ is a non-negative weight. $\mathbf{T}$ is a vector of first order vector fields, $f \in L^2(\Omega)$, and $\mathbf{H}$, F are coefficient functions that are members of the function space determined by the dual of the gain factor in each of the following types of Sobolev inequality, which hold for all $\varphi \in Lip_0(\Omega)$ and that we use as hypotheses:

$$\left(\int_\Omega |\varphi|^2\, w dx\right)^{\frac{1}{2}} \leq C\left(\int_\Omega |\sqrt{Q}\nabla\varphi|^2\, dx\right)^{\frac{1}{2}} \tag{0.2}$$

or

$$\|\varphi\|_{L^\Psi(\Omega;w)} \leq C\left(\int_\Omega |\sqrt{Q}\nabla\varphi|^2\, dx\right)^{\frac{1}{2}} \tag{0.3}$$

where $C > 0$ and $\Psi(t) = t^2\log(e-1+t)^q$ for some $q > 0$. Specifically, the roughness of the geometry associated to the vector fields defined by Q restricts $\mathbf{H}$ and F to be bounded when we assume (0.2), and $\mathbf{H}$ and F can be exponentially integrable if we instead assume that (0.3) holds. Indeed, in the former case, $\mathbf{H}, F \in L^\infty(\Omega; w)$ and in the latter case, $\mathbf{H} \in L^{\Psi_{\mathrm{H}}}(\Omega; w)$ and $F \in L^{\Psi_{\mathrm{F}}}(\Omega; w)$ where $\Psi_{\mathrm{H}}(t) = e^{t^{\frac{2}{\gamma}}} - 1$ and $\Psi_F(t) = e^{t^{\frac{1}{\gamma}}} - 1$ for $0 < \gamma \leq \frac{q}{2}$.

Acknowledgements

I would like to thank my supervisor, Dr. Scott Rodney, for his constant support and guidance throughout the completion of this thesis and inspiring me to continue my studies in mathematics.

Additionally, I would like to thank my committee member, Dr. Ryan Gibara, for many fruitful conversations and careful criticisms of my work.

Lastly, I would like to thank my friends and family for their support throughout my studies. Although they never quite understood what I was doing, they could always see how much it excited me.

Contents

Notation

Symbol	Description
$\mathbb{R}^n$	$n-$dimensional Euclidean space
Ω	An open, connected, bounded subset of $\mathbb{R}^n$
$\partial\Omega$	The boundary of the set Ω
$\mathcal{S}_n$	The collection of all $n \times n$ symmetric, non-negative definite matrices
$\|f\|^p_{L^p(\Omega;w)}$	$\displaystyle\int_\Omega \|f\|^p \, w dx$
$\|f\|^p_{\mathcal{L}^p(Q;\Omega)}$	$\displaystyle\int_\Omega \|\sqrt{Q}\nabla f\|^p \, dx$
$\|f\|_{L^\Psi(\Omega;w)}$	$\inf\Big\{\lambda > 0 : \displaystyle\int_\Omega \Psi\left(\frac{\|f\|}{\lambda}\right) w dx \leq 1\Big\}$
$supp(f)$	The support of a function f; the closure of the set where f is non-zero
$Lip_0(\Omega)$	The collection of Lipshitz functions with compact support in Ω
$QH^1_0(\Omega; w)$	The degenerate Sobolev Space
$\mathbb{1}_E$	$\mathbb{1}_E = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$
$\|\mathbf{f}\|$	$\|\mathbf{f}\| = \left(\displaystyle\sum_{i=1}^N \|f_i\|^2\right)^{\frac{1}{2}}$ for $N \geq 1$

1 Introduction

In the study of partial differential equations, the existence of well-posed classical solutions is often very difficult to prove in practice. Indeed, with the incorporation of data functions having discontinuities or singularities, classical solutions may not exist. However, there may exist a weaker class of solutions if continuity or differentiability is not initially imposed. Indeed, the existence of weak solutions can be studied using techniques in functional analysis that rely on *a priori* estimates that depend only on the norms of given data functions in place of their smoothness properties.

In this thesis, we study a second-order, linear partial differential equation on an open, connected, bounded set $\Omega \subset \mathbb{R}^n$ with degenerate principal part. The most general form of these equations is written as follows, where dependence on $x \in \Omega$ is suppressed:

$$-\frac{1}{w}Div(Q\nabla u) + \mathbf{HR}u + \mathbf{S}'\mathbf{G}u + Fu = f + \mathbf{T}'\mathbf{g}.$$

The coefficient on the principal part $Q : \Omega \to \mathcal{S}_n$ is a matrix-valued function whose entries are Lebesgue measurable scalar functions that are finite almost everywhere. The non-negative weight $w \in L^1_{loc}(\Omega)$ is a measurable function defined on Ω such that $0 \leq w(x) < \infty$ and

$$|Q(x)|_{\text{op}} \leq cw(x) \tag{1.1}$$

for some constant $c > 0$ and for almost every $x \in \Omega$. Note,

$$|Q(x)|_{\text{op}} = \sup\{|Q(x)\xi| : \xi \in \mathbb{R}^n, |\xi| = 1\}.$$

For some $N \in \mathbb{N}$, $\mathbf{R}$ is an N-tuple of first order vector fields, $\mathbf{S}$', $\mathbf{T}$' are adjoints of first order vector fields, $f, F \in L^2(\Omega; w)$, and $\mathbf{G}, \mathbf{H}, \mathbf{g} \in L^2(\Omega; w)$; see [1] for explicit details.

As the general existence theory for partial differential equations of this form is currently in development, this thesis considers an edited problem. Specifically,

$$\begin{cases} -\frac{1}{w}\text{Div}\,(Q\nabla u) + \mathbf{HT}u + Fu &= f, \quad x \in \Omega; \\ u &= 0, \quad x \in \partial\Omega; \end{cases} \tag{1.2}$$

where $f \in L^2(\Omega)$, $\mathbf{T}$ is a vector of first order vector fields, and membership of the functions $\mathbf{H}$ and F is dependent on the type of Sobolev inequality that is available. Hereafter, $Q : \Omega \to \mathcal{S}_n$ is a matrix-valued function and may have determinant that vanishes at a finite number of points. This behavior is referred to as degeneracy. The matrix Q is diagonalizable and so,

$$Q(x) = U^T(x)D(x)U(x)$$

almost everywhere in Ω where U is a unitary matrix-valued function on Ω and

$$D(x) = \text{diag}(\lambda_1(x), ..., \lambda_n(x))$$

is a diagonal matrix. Since Q is measurable, U and the eigenvalues λ_i can be chosen to be measurable and we note $0 \leq \lambda_i(x) < \infty$ for all i and for almost every $x \in \Omega$; see [2]. Therefore, we may define $\sqrt{Q} = U^T\sqrt{D}\,U$ where $\sqrt{D} = \text{diag}(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})$. Since w satisfies (1.1), $|Q(x)|_{\text{op}} \in L^1_{loc}(\Omega)$ and consequently, the entries of Q are in $L^1_{loc}(\Omega)$. The primary objective of this thesis is to prove the existence of weak solutions to (1.2) in the degenerate Sobolev space $QH^1_0(\Omega; w)$. In particular, we prove the following two theorems:

Theorem 1.1. *Let Q be a non-negative definite symmetric matrix-valued function in $\Omega \subset \mathbb{R}^n$ with the non-negative weight $w \in L^1_{loc}(\Omega)$ that satisfies* (1.1)*. Suppose there exists a constant $C > 0$ such that*

$$\|\varphi\|_{L^2(\Omega;w)} \leq C\|\sqrt{Q}\nabla\varphi\|_{L^2(\Omega)}$$

for all $\varphi \in Lip_0(\Omega)$. Then, given $f \in L^2(\Omega)$, there is a unique weak solution $(u, \nabla u) \in QH^1_0(\Omega; w)$ of the Dirichlet problem

$$\begin{cases} -\dfrac{1}{w}Div(Q\nabla u) + \mathbf{HT}u + Fu &= f, \quad x \in \Omega; \\ u &= 0, \quad x \in \partial\Omega; \end{cases} \tag{1.3}$$

whenever $\mathbf{H}, F \in L^\infty(\Omega; w)$ and $\mathbf{T}$ is a vector of first order vector fields satisfying (4.2)*.*

Theorem 1.2. *Let Q be a non-negative definite symmetric matrix-valued function in $\Omega \subset \mathbb{R}^n$ with the non-negative weight $w \in L^1_{loc}(\Omega)$ that satisfies* (1.1)*. Suppose there exists a constant $C > 0$ such that*

$$\|\varphi\|_{L^\Psi(\Omega;w)} \leq C\|\sqrt{Q}\nabla\varphi\|_{L^2(\Omega)}$$

for all $\varphi \in Lip_0(\Omega)$ *where* $\Psi(t) = t^2 \log(e-1+t)^q$ *for* $q > 0$. *Then, given* $f \in L^2(\Omega)$, *there is a unique weak solution* $(u, \nabla u) \in QH_0^1(\Omega; w)$ *of the Dirichlet problem*

$$\begin{cases} -\frac{1}{w} Div(Q\nabla u) + \mathbf{HT}u + Fu &= f, \quad x \in \Omega; \\ u &= 0, \quad x \in \partial\Omega; \end{cases} \tag{1.4}$$

whenever $\mathbf{H} \in L^{\Psi_H}(\Omega; w), F \in L^{\Psi_F}(\Omega; w)$ *where* $\Psi_H(t) = e^{t^{\frac{2}{\gamma}}} - 1$, $\Psi_F(t) = e^{t^{\frac{1}{\gamma}}} - 1$ *for* $0 < \gamma \leq \frac{q}{2}$ *and* $\mathbf{T}$ *is a vector of first order vector fields satisfying* (4.2).

The thesis is organized as follows. In Chapter 2, we provide preliminary results about $L^p(\Omega; w)$ spaces, Orlicz spaces, and the degenerate Sobolev space $QH_0^1(\Omega; w)$. In Chapter 3, we prove the Lax-Milgram Theorem and the Fredholm Alternative as they are fundamental to our main results. Finally, in Chapter 4, we prove Theorem 1.1 and Theorem 1.2. Supplementary material is provided in the Appendices.

2 Preliminaries

2.1 $L^p(\Omega; w)$ Spaces

Definition 2.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose that $1 \leq p < \infty$. The space $L^p(X, \mathcal{A}, \mu)$ is defined to be the collection of equivalence classes of μ-measurable functions $f : X \to \mathbb{R}$ for which*

$$\|f\|_{L^p(X,\mathcal{A},\mu)} = \left(\int_X |f(x)|^p \, d\mu \right)^{\frac{1}{p}} < \infty.$$

Functions $f, g \in L^p(X, \mathcal{A}, \mu)$ are equivalent if and only if $\|f - g\|_{L^p(X,\mathcal{A},\mu)} = 0$.

Definition 2.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space. The essential supremum of a given μ-measurable function $f : X \to \mathbb{R}$ is defined as*

$$\operatorname*{ess\,sup}_X f = \inf\{\alpha \in \mathbb{R} : \mu(\{x \in X : |f(x)| > \alpha\}) = 0\}.$$

The space $L^\infty(X, \mathcal{A}, \mu)$ is the collection of equivalence classes of μ-measurable functions f for which

$$\|f\|_{L^\infty(X,\mathcal{A},\mu)} = \operatorname*{ess\,sup}_X |f| < \infty.$$

Functions $f, g \in L^\infty(X, \mathcal{A}, \mu)$ are equivalent if and only if $\|f-g\|_{L^\infty(X,\mathcal{A},\mu)} = 0$.

In this study, X is the open, connected, bounded domain $\Omega \subset \mathbb{R}^n$. The measure μ is either the Lebesgue measure dx or the weighted Lebesgue measure $w\,dx$ on the Lebesgue measurable subsets $\mathcal{M}$ of Ω. To avoid confusion in the development hereafter, we write $\|f\|_{L^p(\Omega,\mathcal{M},w)} = \|f\|_{L^p(\Omega;w)}$. An important estimate that we call upon frequently is Hölder's inequality. This is very well-documented in the literature, however, we provide the proof in Theorem 2.5 for the benefit of the reader. To prove Theorem 2.5, we first need to establish Young's inequality as detailed in Proposition 2.3. Note, after the establishment of Theorem 2.6, we conclude this section by proving three immediate consequences of Hölder's inequality.

Proposition 2.3. *Let $1 < p \leq q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any a, $b \geq 0$,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.1}$$

Proof. Let $1 < p \leq q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Note, if $a = 0$ or $b = 0$ then there is nothing to prove. Let $a, b > 0$ and consider the equivalent form of (2.1):

$$\frac{ab}{b^q} - \frac{a^p}{pb^q} \leq \frac{1}{q}.$$

To prove this inequality, it suffices to show

$$\sup\left\{\frac{ab}{b^q} - \frac{a^p}{pb^q} : a > 0, b > 0\right\} = \frac{1}{q}. \tag{2.2}$$

Performing elementary algebra gives

$$\frac{ab}{b^q} - \frac{a^p}{pb^q} = \frac{a}{b^{q-1}} - \frac{1}{p}\left(\frac{a^p}{b^q}\right) = \left(\frac{a^p}{b^q}\right)^{\frac{1}{p}} - \frac{1}{p}\left(\frac{a^p}{b^q}\right)$$

since $\frac{1}{p} + \frac{1}{q} = 1$. Setting $\xi = \dfrac{a^p}{b^q} > 0$, we consider $f(\xi) = \xi^{\frac{1}{p}} - \frac{1}{p}\xi$ on $(0, \infty)$. Since $\lim\limits_{\xi \to 0^+} f(\xi) = 0$ and $\lim\limits_{\xi \to \infty} f(\xi) = -\infty$, the supremum of $f(\xi)$ is either 0 or occurs at a critical point. Now, $f'(\xi) = \frac{1}{p}\left(\xi^{-\frac{1}{q}} - 1\right)$ as $\frac{1}{p} - 1 = -\frac{1}{q}$ where $f'(\xi) = 0$ occurs at $\xi = 1$. Notice, $f(1) = 1 - \frac{1}{p} = \frac{1}{q} > 0$ and $f''(\xi) = -\dfrac{1}{pq}\xi^{-\frac{1}{q}-1} < 0$ so $\xi = 1$ is an absolute maximum, proving (2.2).

□

Corollary 2.4. *Let $1 < p \leq q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $a, b \geq 0$ and $\varepsilon > 0$,*

$$ab \leq \frac{\varepsilon a^p}{p} + \frac{\varepsilon^{-\frac{q}{p}} b^q}{q} \leq \varepsilon a^p + \varepsilon^{-\frac{q}{p}} b^q. \tag{2.3}$$

Proof. Let $\varepsilon > 0$ and write $ab = \varepsilon^{\frac{1}{p}} a \dfrac{b}{\varepsilon^{\frac{1}{p}}}$. Then, by Proposition 2.3

$$ab = \left(\varepsilon^{\frac{1}{p}} a\right)\left(\varepsilon^{-\frac{1}{p}} b\right) \leq \frac{\varepsilon a^p}{p} + \frac{\varepsilon^{-\frac{q}{p}} b^q}{q} \leq \varepsilon a^p + \varepsilon^{-\frac{q}{p}} b^q.$$

□

Theorem 2.5. *Let $f,\ g : \Omega \to \mathbb{R}$ and let $1 < p \leq \infty, 1 \leq q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\Omega; w)$ and $g \in L^q(\Omega; w)$, then $fg \in L^1(\Omega; w)$ with*

$$\Big|\int_\Omega f(x)g(x)\, w dx\Big| \leq \|f\|_{L^p(\Omega;w)}\|g\|_{L^q(\Omega;w)}. \tag{2.4}$$

Proof. Let $1 < p \leq \infty, 1 \leq q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^p(\Omega; w)$ and $g \in L^q(\Omega; w)$. Notice, if f or g equal 0 for almost every $x \in \Omega$, then there is nothing to prove. So, we proceed assuming $f, g \neq 0$ almost everywhere in Ω. Suppose $p = \infty$ and $q = 1$. By definition 2.2, $|f(x)| \leq \|f\|_{L^\infty(\Omega;w)}$ for almost every $x \in \Omega$ and so

$$\int_\Omega |f(x)g(x)|\, w dx \leq \int_\Omega \|f\|_{L^\infty(\Omega;w)}\, |g(x)|\, w dx = \|f\|_{L^\infty(\Omega;w)}\|g\|_{L^1(\Omega;w)}.$$

Suppose now that $p < \infty$ and consider the instance when

$$\|f\|_{L^p(\Omega;w)} = \|g\|_{L^q(\Omega;w)} = 1.$$

For each $x \in \Omega$, set $a = |f(x)|$ and $b = |g(x)|$. Since $a, b \geq 0$, Proposition 2.3 gives,

$$|f(x)||g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q}.$$

Integrating both sides we find,

$$\begin{aligned}
\Big|\int_\Omega f(x)g(x)\, w dx\Big| &\leq \int_\Omega |f(x)g(x)|\, w dx \\
&\leq \int_\Omega \frac{|f(x)|^p}{p}\, w dx + \int_\Omega \frac{|g(x)|^q}{q}\, w dx \\
&= \frac{1}{p}\|f\|^p_{L^p(\Omega;w)} + \frac{1}{q}\|g\|^q_{L^q(\Omega;w)} \\
&= 1.
\end{aligned}$$

Consider $\Big|\int_\Omega \alpha(x)\beta(x)\, w dx\Big|$ where $\alpha(x) = \dfrac{f(x)}{\|f\|_{L^p(\Omega;w)}}$, $\beta(x) = \dfrac{g(x)}{\|g\|_{L^q(\Omega;w)}}$. Then, $\|\alpha\|_{L^p(\Omega;w)} = \|\beta\|_{L^q(\Omega;w)} = 1$ and our previous estimate ensures

$$\Big|\int_\Omega \frac{f(x)}{\|f\|_{L^p(\Omega;w)}} \frac{g(x)}{\|g\|_{L^q(\Omega;w)}}\, w dx\Big| \leq 1.$$

Equivalently,

$$\Big|\int\limits_{\Omega} f(x)g(x)\,wdx\Big| \leq \|f\|_{L^p(\Omega;w)}\|g\|_{L^q(\Omega;w)}.$$

□

Note, this result can be generalized to functions $f_1, f_2, ..., f_n$ in spaces $L^{p_1}(\Omega;w), L^{p_2}(\Omega;w), ..., L^{p_n}(\Omega;w)$ respectively as described in the following theorem.

Theorem 2.6. *Let $f_i \in L^{p_i}(\Omega;w)$ for $i = 1, ..., n$ and $\dfrac{1}{p_1} + ... + \dfrac{1}{p_n} = 1$. Then,*

$$\Big|\int_{\Omega} \prod_{i=1}^{n} f_i(x)\,wdx\Big| \leq \prod_{i=1}^{n} \|f_i\|_{L^{p_i}(\Omega;w)}. \tag{2.5}$$

Proof. For the base case, set $n = 2$. This result was proved in Theorem 2.5 and so, we proceed with the induction hypothesis. Suppose that

$$\Big|\int_{\Omega} \prod_{i=1}^{n} f_i(x)\,wdx\Big| \leq \prod_{i=1}^{n} \|f_i\|_{L^{p_i}(\Omega;w)}$$

holds for n measurable functions where $n \geq 2$ whenever $\displaystyle\sum_{i=1}^{n} \frac{1}{p_i} = 1$. Take $n+1$ functions such that $\displaystyle\sum_{i=1}^{n+1} \frac{1}{p_i} = 1$ and consider

$$\begin{aligned}
\Big|\int_{\Omega} \prod_{i=1}^{n+1} f_i(x)\,wdx\Big| &= \Big|\int_{\Omega} \prod_{i=2}^{n+1} f_i(x) f_1(x)\,wdx\Big| \\
&\leq \Big\|\prod_{i=2}^{n+1} f_i\Big\|_{L^{\frac{p_1}{p_1-1}}(\Omega;w)} \|f_1\|_{L^{p_1}(\Omega;w)} \\
&= \left(\int_{\Omega} \prod_{i=2}^{n+1} |f_i(x)|^{\frac{p_1}{p_1-1}}\,wdx\right)^{\frac{p_1-1}{p_1}} \|f_1\|_{L^{p_1}(\Omega;w)}.
\end{aligned}$$

Since $\displaystyle\sum_{i=1}^{n+1} \frac{1}{p_i} = 1$ then,

$$\sum_{i=2}^{n+1} \frac{1}{p_i} = 1 - \frac{1}{p_1}.$$

Setting $\alpha = \dfrac{p_1}{p_1 - 1}$ and defining $\gamma_i = \dfrac{p_i}{\alpha}$ for $i = 2, ..., n+1$ then,

$$\sum_{i=2}^{n+1} \frac{1}{\gamma_i} = \sum_{i=2}^{n+1} \frac{\alpha}{p_i} = 1.$$

Employing the induction hypothesis, we can further estimate using our previous work. Specifically,

$$\begin{aligned}\left(\int_\Omega \prod_{i=2}^{n+1} |f_i(x)|^\alpha \, w dx \right) &\leq \prod_{i=2}^{n+1} \|f_i^\alpha\|_{L^{\gamma_i}(\Omega;w)} \\ &= \prod_{i=2}^{n+1} \|f_i\|^\alpha_{L^{p_i}(\Omega;w)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left| \int_\Omega \prod_{i=1}^{n+1} f_i(x) \, w dx \right| &\leq \prod_{i=2}^{n+1} \left(\|f_i\|^\alpha_{L^{p_i}(\Omega;w)} \right)^{\frac{1}{\alpha}} \|f_1\|_{L^{p_1}(\Omega;w)} \\ &= \prod_{i=1}^{n+1} \|f_i\|_{L^{p_i}(\Omega;w)}.\end{aligned}$$

By the principle of mathematical induction, (2.5) holds for every $n \in \mathbb{N}$.

□

Lemma 2.7. *Suppose $w(\Omega) < \infty$ and $1 \leq p < \infty, 1 < q \leq \infty$ where $p \leq q$ and $f \in L^q(\Omega; w)$. Then,*

$$w(\Omega)^{-\frac{1}{p}} \|f\|_{L^p(\Omega;w)} \leq w(\Omega)^{-\frac{1}{q}} \|f\|_{L^q(\Omega;w)}.$$

Proof. If $p = q$, then there is nothing to prove. If $1 \leq p < q$ and $q = \infty$, then for almost every $x \in \Omega$,

$$\|f\|^p_{L^p(\Omega;w)} = \int_\Omega |f(x)|^p \cdot 1 \, w dx \leq \|f\|^p_{L^\infty(\Omega;w)} \int_\Omega w dx = \|f\|^p_{L^\infty(\Omega;w)} w(\Omega).$$

Taking the *pth* root of both side gives

$$w(\Omega)^{-\frac{1}{p}} \|f\|_{L^p(\Omega;w)} \leq \|f\|_{L^\infty(\Omega;w)}.$$

Suppose now $1 \leq p < \infty$ and $1 < q < \infty$ such that $p < q$. Set $p' = \frac{q}{p}$, $q' = \frac{q}{q-p}$, then $\frac{1}{p'} + \frac{1}{q'} = 1$ and so, by Theorem 2.5

$$\|f\|^p_{L^p(\Omega;w)} = \int_\Omega |f(x)|^p \cdot 1 \, w dx \leq \left(\int_\Omega |f(x)|^{p \cdot \frac{q}{p}} \, w dx \right)^{\frac{p}{q}} \left(\int_\Omega |1|^{\frac{q}{q-p}} \, w dx \right)^{\frac{q-p}{q}}.$$

Since $\left(\int_\Omega |1|^{\frac{q}{q-p}} \, w dx \right)^{\frac{q-p}{q}} = w(\Omega)^{\frac{q-p}{q}}$, taking the pth root of both sides gives

$$\begin{aligned} \|f\|_{L^p(\Omega;w)} &\leq \|f\|_{L^q(\Omega;w)} w(\Omega)^{\frac{q-p}{pq}} \\ &= \|f\|_{L^q(\Omega;w)} w(\Omega)^{\frac{1}{p}-\frac{1}{q}}. \end{aligned}$$

□

Lemma 2.8. *Let $f \in L^r(\Omega; w)$, $1 \leq p \leq q < \infty$, $q \leq r \leq \infty$, and $\dfrac{1}{q} = \dfrac{\lambda}{p} + \dfrac{1-\lambda}{r}$ for $\lambda \in (0,1)$. Then,*

$$\|f\|_{L^q(\Omega;w)} \leq \|f\|^\lambda_{L^p(\Omega;w)} \|f\|^{1-\lambda}_{L^r(\Omega;w)}.$$

Proof. Let $\lambda \in (0,1)$ and set $\dfrac{1}{q} = \dfrac{\lambda}{p} + \dfrac{1-\lambda}{r}$ where $1 \leq p \leq q < \infty$ and $q \leq r \leq \infty$. If $r = \infty$, then $p = \lambda q$ and so

$$\begin{aligned} \|f\|^q_{L^q(\Omega;w)} &= \int_\Omega |f(x)|^q \, w dx \\ &= \int_\Omega |f(x)|^{q\lambda} \, |f(x)|^{q(1-\lambda)} \, w dx \\ &\leq \int_\Omega |f(x)|^p \|f\|^{q(1-\lambda)}_{L^\infty(\Omega;w)} \, w dx \\ &= \|f\|^p_{L^p(\Omega;w)} \|f\|^{q(1-\lambda)}_{L^\infty(\Omega;w)}. \end{aligned}$$

Taking the qth root of both sides gives

$$\|f\|_{L^q(\Omega;w)} \leq \|f\|^\lambda_{L^p(\Omega;w)} \|f\|^{1-\lambda}_{L^\infty(\Omega;w).}$$

Suppose now $r < \infty$ and notice $\dfrac{q\lambda}{p} + \dfrac{q\,(1-\lambda)}{r} = 1$. Then, Theorem 2.5 gives

$$\begin{aligned} \|f\|^q_{L^q(\Omega;w)} &\leq \|f^{q\lambda}\|_{L^{\frac{p}{q\lambda}}(\Omega;w)} \|f^{q(1-\lambda)}\|_{L^{\frac{r}{q(1-\lambda)}}(\Omega;w)} \\ &= \left(\int_\Omega |f(x)|^{q\lambda \cdot \frac{p}{q\lambda}} \, w dx \right)^{\frac{q\lambda}{p}} \left(\int_\Omega |f(x)|^{q(1-\lambda) \cdot \frac{r}{q(1-\lambda)}} \, w dx \right)^{\frac{q(1-\lambda)}{r}} \\ &= \left(\int_\Omega |f(x)|^p \, w dx \right)^{\frac{q\lambda}{p}} \left(\int_\Omega |f(x)|^r \, w dx \right)^{\frac{q(1-\lambda)}{r}}. \end{aligned}$$

Taking the *qth* root of both sides, it follows that

$$\|f\|_{L^q(\Omega;w)} \leq \|f\|^{\lambda}_{L^p(\Omega;w)} \|f\|^{1-\lambda}_{L^r(\Omega;w)}.$$

□

Lemma 2.9. *Let* $f \in L^r(\Omega; w)$, $1 \leq p \leq q < \infty$, $q < r \leq \infty$ *such that* $\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$ *with* $\lambda \in (0,1)$. *Then,*

$$\|f\|_{L^q(\Omega;w)} \leq \varepsilon \|f\|_{L^r(\Omega;w)} + \varepsilon^{-\mu} \|f\|_{L^p(\Omega;w)}$$

for any $\varepsilon > 0$, *and* $\mu = \left(\frac{1}{p} - \frac{1}{q}\right) \Big/ \left(\frac{1}{q} - \frac{1}{r}\right)$.

Proof. Combining inequalities from Lemma 2.8 and Corollary 2.4, we find

$$\begin{aligned} \|f\|_{L^q(\Omega;w)} &\leq \|f\|^{\lambda}_{L^p(\Omega;w)} \|f\|^{1-\lambda}_{L^r(\Omega;w)} \\ &\leq \varepsilon \|f\|^{(1-\lambda)s}_{L^r(\Omega;w)} + \varepsilon^{-\frac{t}{s}} \|f\|^{\lambda t}_{L^p(\Omega;w)} \end{aligned}$$

with $s = \dfrac{1}{1-\lambda} > 1$ and $t = \dfrac{1}{\lambda} > 1$ as $\lambda \in (0,1)$. Note, $\dfrac{1}{s} + \dfrac{1}{t} = 1$ where $\dfrac{t}{s} = \dfrac{1}{\lambda} \cdot (1-\lambda)$. Setting $\dfrac{1}{q} = \dfrac{\lambda}{p} + \dfrac{1-\lambda}{r}$, we find

$$\lambda = \frac{\left(\frac{1}{q} - \frac{1}{r}\right)}{\left(\frac{1}{p} - \frac{1}{r}\right)}; \qquad 1-\lambda = \frac{\left(\frac{1}{p} - \frac{1}{q}\right)}{\left(\frac{1}{p} - \frac{1}{r}\right)}; \qquad \frac{t}{s} = \frac{1-\lambda}{\lambda} = \frac{\left(\frac{1}{p} - \frac{1}{q}\right)}{\left(\frac{1}{q} - \frac{1}{r}\right)} = \mu.$$

Therefore,

$$\|f\|_{L^q(\Omega;w)} \leq \varepsilon \|f\|_{L^r(\Omega;w)} + \varepsilon^{-\mu} \|f\|_{L^p(\Omega;w)}.$$

□

2.2 Orlicz Spaces

Orlicz function spaces are a generalization of Lebesgue spaces. As opposed to Lebesgue spaces which are associated to a parameter $p \in [1, \infty)$, Orlicz spaces are defined in terms of Young functions. In order to provide a comprehensive characterization of Young functions, we begin by defining their density. Let

$a : [0, \infty) \to [0, \infty)$ be a right continuous, non-decreasing function such that $a(0) = 0, a(t) > 0$ for $t \in (0, \infty)$, and $a(t) \to \infty$ as $t \to \infty$. Given a density function $a(t)$, we set

$$A(t) = \int_0^t a(s)\, ds.$$

Such $A(t)$ are called Young functions and satisfy the following definition; see [3].

Definition 2.10. *If $A(t)$ is a Young function, then*

i) $A(t)$ is convex, continuous, and strictly increasing.

ii) $A(0) = 0, \dfrac{A(t)}{t} \to \infty$ as $t \to \infty$.

iii) For $t > 0$, $A'(t) = a(t)$ at every point of continuity of a (except on an at most countable set).

Definition 2.11. *Let $(X, \mathcal{A}, \mu)$ be a measure space and let Ψ be a Young function. The Orlicz class $L^{\Psi}(X, \mathcal{A}, \mu)$ is collection of equivalence classes of μ-measurable function $f : X \to \mathbb{R}$ for which the Luxembourg norm*

$$\|f\|_{L^{\Psi}(X, \mathcal{A}, \mu)} = \inf\Big\{\lambda > 0 : \int_X \Psi\left(\frac{|f|}{\lambda}\right) d\mu \leq 1\Big\} < \infty. \tag{2.6}$$

Functions $f, g \in L^{\Psi}(X, \mathcal{A}, \mu)$ are equivalent if and only if $\|f - g\|_{L^{\Psi}(X, \mathcal{A}, \mu)} = 0$.

Note, the Orlicz class $L^{\Psi}(X, \mathcal{A}, \mu)$ is a complete space; see [3]. To justify Orlicz spaces are a generalization of Lebesgue spaces, we demonstrate $\|f\|_{L^{\Psi}(X, \mathcal{A}, \mu)} = \|f\|_{L^p(X, \mathcal{A}, \mu)}$ when $\Psi(t) = t^p$ for $p > 1$. Indeed, for $f \in L^{\Psi}(X, \mathcal{A}, \mu)$,

$$\begin{aligned}
\|f\|_{L^{\Psi}(X, \mathcal{A}, \mu)} &= \inf\Big\{\lambda > 0 : \int_X \Psi\left(\frac{|f(x)|}{\lambda}\right) d\mu \leq 1\Big\} \\
&= \inf\Big\{\lambda > 0 : \int_X \frac{|f(x)|^p}{\lambda^p}\, d\mu \leq 1\Big\} \\
&= \inf\Big\{\lambda > 0 : \int_X |f(x)|^p\, d\mu \leq \lambda^p\Big\} \\
&= \inf\Big\{\lambda > 0 : \left(\int_X |f(x)|^p\, d\mu\right)^{\frac{1}{p}} \leq \lambda\Big\} \\
&= \|f\|_{L^p(X, \mathcal{A}, \mu)}.
\end{aligned}$$

The Orlicz spaces of focus in this work are characterized by Young functions of the form

$$\Psi(t) = t^2 \log(e - 1 + t)^q, \tag{2.7}$$

where $q > 0$. This type of Young function is referred to as a 'log-bump' function as it provides a scale of Orlicz classes that lie between Lebesgue spaces. Analogous to the previous section, Hölders inequality on this scale provides an important estimate in our later work. Indeed, the generalization of this inequality is required to establish the interpolation inequality seen in Lemma 2.22. To prove Hölders inequality on this scale, we first provide the reader with fundamental definitions followed by, without proof, Youngs's inequality and Fatou's Lemma. Note, these proofs can be found in [3] and [4]. We then include the proof of an auxiliary lemma before proving Hölders inequality in Theorem 2.18.

Definition 2.12. *Given a non-decreasing function a, we define its generalized right inverse to be the function*

$$\bar{a}(t) = \sup\{s : a(s) \leq t\} \tag{2.8}$$

for all $t \geq 0$.

Definition 2.13. *Given a Young function $A(t)$, if we define*

$$\bar{A}(t) = \int_0^t \bar{a}(s)\, ds, \tag{2.9}$$

then $\bar{A}(t)$ is the complementary Young Function function of $A(t)$; see [3].

Example 2.14. Let $A(t) = t^p$ for $1 < p < \infty$ then $a(t) = pt^{p-1}$. So,

$$\bar{a}(t) = \sup\{s : ps^{p-1} \leq t\} = \left(\frac{1}{p}t\right)^{\frac{1}{p-1}}.$$

Then,

$$\bar{A}(t) = \frac{1}{p^{\frac{1}{p-1}}} \int_0^t s^{\frac{1}{p-1}}\, ds = c_p \frac{t^{\frac{p}{p-1}}}{q}.$$

Theorem 2.15. *Let $A(t)$ and $\bar{A}(t)$ be a complementary pair of Young functions. Then for all $s, t \geq 0$,*

$$st \leq A(s) + \bar{A}(t). \tag{2.10}$$

Lemma 2.16. *If $\{f_n\}$ is a sequence of non-negative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a measurable set Θ then*

$$\int_\Theta f(x)\, w dx \leq \liminf_{n\to\infty} \int_\Theta f_n(x)\, w dx.$$

Lemma 2.17. *Suppose A is a Young function. Then, $\|f\|_{L^A(\Omega;w)} \leq 1$ if and only if*

$$\int_\Omega A(|f(x)|)\, w dx \leq 1.$$

Proof. The following argument was adapted from [3]. Suppose $\|f\|_{L^A(\Omega;w)} \leq 1$. Then, given any $\lambda > 1$,

$$\int_\Omega A\left(\frac{|f(x)|}{\lambda}\right) w dx \leq 1.$$

Define a decreasing sequence $\lambda_k > \lambda_{k+1} > \ldots > 1$ such that $\lim_{k\to\infty} \lambda_k = 1$. Fix $x \in \Omega$. Then, by the continuity of A,

$$\lim_{k\to\infty} A\left(\frac{|f(x)|}{\lambda_k}\right) = A\left(\lim_{k\to\infty} \frac{|f(x)|}{\lambda_k}\right) = A(|f(x)|).$$

By Lemma 2.16, we have that

$$\begin{aligned}\int_\Omega A(|f(x)|)\, w dx &= \int_\Omega \lim_{k\to\infty} A\left(\frac{|f(x)|}{\lambda_k}\right) w dx \\ &\leq \liminf_{k\to\infty} \int_\Omega A\left(\frac{|f(x)|}{\lambda_k}\right) w dx \\ &\leq 1\end{aligned}$$

since $\int_\Omega A\left(\frac{|f(x)|}{\lambda_k}\right) w dx \leq 1$ for each k.

Suppose now that

$$\int_\Omega A(|f(x)|)\, w dx \leq 1.$$

Then, $1 \in \left\{\lambda > 0 : \int_\Omega A\left(\frac{|f(x)|}{\lambda}\right) w dx \leq 1\right\}$ and so, $\|f\|_{L^A(\Omega;w)} \leq 1$.

□

Theorem 2.18. *Let A and $\bar{A}$ be complementary Young functions. Let f, g be such that $f \in L^A(\Omega; w)$ and $g \in L^{\bar{A}}(\Omega; w)$. Then,*

$$\int_\Omega |f(x)g(x)|\, w dx \leq 2\|f\|_{L^A(\Omega;w)}\|g\|_{L^{\bar{A}}(\Omega;w)}. \tag{2.11}$$

Proof. The following proof was adpated from [3]. Let A and $\bar{A}$ be Young functions and let f, g be such that $f \in L^A(\Omega; w)$ and $g \in L^{\bar{A}}(\Omega; w)$. Notice, if f or g equal 0 almost everywhere in Ω, then there is nothing to prove. So, we proceed under the assumption $\|f\|_{L^A(\Omega;w)}, \|g\|_{L^{\bar{A}}(\Omega;w)} > 0$. Let $\|f\|_{L^A(\Omega;w)} = \|g\|_{L^{\bar{A}}(\Omega;w)} = 1$. Then, Lemma 2.17 gives

$$\int_\Omega A(|f(x)|)\, w dx \leq 1;$$
$$\int_\Omega \bar{A}(|g(x)|)\, w dx \leq 1.$$

By Theorem 2.15, we find

$$\int_\Omega |f(x)g(x)|\, w dx \leq \int_\Omega A(|f(x)|)\, w dx + \int_\Omega \bar{A}(|g(x)|)\, w dx \leq 2.$$

For measurable functions α and β, consider $\displaystyle\int_\Omega |\alpha(x)\beta(x)|\, w dx$ where

$$\|\alpha\|_{L^A(\Omega;w)} = \|\beta\|_{L^{\bar{A}}(\Omega;w)} = 1.$$

Setting $\alpha(x) = \dfrac{f(x)}{\|f\|_{L^A(\Omega;w)}}$ and $\beta(x) = \dfrac{g(x)}{\|g\|_{L^{\bar{A}}(\Omega;w)}}$, our previous estimate ensures

$$\int_\Omega |f(x)g(x)|\, w dx \leq 2\|f\|_{L^A(\Omega;w)}\|g\|_{L^{\bar{A}}(\Omega;w)}.$$

□

This result can be generalized to encompass m measurable Young functions on an arbitrary measure space $(X, \mathcal{A}, \mu)$ as detailed in Appendix A Theorem 5.1. Notice the increased complexity and additional intermediary estimates used to prove Theorem 5.1 as compared to Theorem 2.6. Indeed, this continues throughout our work as evident when comparing the analyses in subsection 4.2.1 to 4.2.2. As a consequence, we require the interpolation inequality seen in Lemma 2.22 and Propositions 2.23, 2.24. The proof of Lemma 2.22 depends on the following proposition, known as the rescaling property of the Luxembourg norm, and lemma.

Proposition 2.19. *Given $1 < p < \infty$ and Young functions Γ, Ψ such that $\Gamma(t) = \Psi(t^{\frac{1}{p}})$. Then,*

$$\| \, |f|^p \, \|_{L^\Gamma(\Omega;w)} = \|f\|^p_{L^\Psi(\Omega;w)}. \tag{2.12}$$

Proof. Let $1 < p < \infty$ and $\lambda > 0$ be given. Set $\lambda^{\frac{1}{p}} = \mu$. Then, for $\Gamma(t) = \Psi(t^{\frac{1}{p}})$,

$$\begin{aligned}
\| \, |f|^p \, \|_{L^\Gamma(\Omega;w)} &= \inf\Big\{\lambda > 0 : \int_\Omega \Gamma\left(\frac{|f(x)|^p}{\lambda}\right) w dx \leq 1\Big\} \\
&= \inf\Big\{\lambda > 0 : \int_\Omega \Psi\left(\frac{|f(x)|}{\lambda^{\frac{1}{p}}}\right) w dx \leq 1\Big\} \\
&= \inf\Big\{\mu^p > 0 : \int_\Omega \Psi\left(\frac{|f(x)|}{\mu}\right) w dx \leq 1\Big\} \\
&= \|f\|^p_{L^\Psi(\Omega;w)}.
\end{aligned}$$

□

Lemma 2.20. *Let Ψ_1, Ψ_2, Φ be Young functions and α_1, α_2 positive real numbers for which*

i) $\Theta_1(t) = \Psi_1(t^{\frac{1}{\alpha_1}})$ and $\Theta_2(t) = \Psi_2(t^{\frac{1}{\alpha_2}})$ are Young functions, and

ii) $(\Psi_1^{-1}(t))^{\alpha_1}(\Psi_2^{-1}(t))^{\alpha_2} \leq \Phi^{-1}(ct)$

for some $c > 0$ and for all $t \geq 0$. Then, there exists a constant $C > 0$ so that for all $f \in L^{\Psi_1}(\Omega; w), g \in L^{\Psi_2}(\Omega; w)$,

$$\| |f|^{\alpha_1} |g|^{\alpha_2} \|_{L^\Phi(\Omega;w)} \leq C \|f\|^{\alpha_1}_{L^{\Psi_1}(\Omega;w)} \|g\|^{\alpha_2}_{L^{\Psi_2}(\Omega;w)}.$$

Proof. Suppose f, g are measurable functions on Ω. Since $\Theta_1(t) = \Psi_1(t^{\frac{1}{\alpha_1}})$ and $\Theta_2(t) = \Psi_2(t^{\frac{1}{\alpha_2}})$ are Young functions with inverses

$$\begin{aligned}
\Theta_1^{-1}(t) &= (\Psi_1^{-1}(t))^{\alpha_1}; \\
\Theta_2^{-1}(t) &= (\Psi_2^{-1}(t))^{\alpha_2},
\end{aligned}$$

hypothesis (ii) ensures that Θ_1, Θ_2, Φ satisfy

$$\Theta_1^{-1}(t)\Theta_2^{-1}(t) \leq \Phi(ct)$$

for some $c > 0$. Therefore, by Theorem 5.1 with $\lambda_1 = \lambda_2 = 1$ and Proposition 2.19,

$$\begin{aligned}\||f|^{\alpha_1}|g|^{\alpha_2}\|_{L^\Phi(\Omega;w)} &\le C\||f|^{\alpha_1}\|_{L^{\Theta_1}(\Omega;w)}\||g|^{\alpha_2}\|_{L^{\Theta_2}(\Omega;w)} \\ &= C\|f\|^{\alpha_1}_{L^{\Psi_1}(\Omega;w)}\|g\|^{\alpha_2}_{L^{\Psi_2}(\Omega;w)}\end{aligned}$$

where C is a constant that depends only on Ψ_1, Ψ_2, and Φ.

□

Remark 2.21. *Although $A(t) = t$ is not a Young function, the following lemma still holds since it generates the Banach space $L^1(\Omega; w)$ with respect to the Luxembourg norm.*

Lemma 2.22. *Let $\lambda \in (0, 1)$ and suppose that Ψ, Φ are Young functions for which*

$$t^{1-\lambda}(\Psi^{-1}(t))^\lambda \le \Phi^{-1}(ct)$$

for all $t \ge 0$. Then, given $\varepsilon > 0$, there is a constant $C = C(\varepsilon, \Psi, \Phi, \lambda)$ so that

$$\|f\|_{L^\Phi(\Omega;w)} \le C\|f\|_{L^1(\Omega;w)} + \varepsilon\|f\|_{L^\Psi(\Omega;w)}$$

for any $f \in L^1(\Omega; w) \cap L^\Psi(\Omega; w)$.

Proof. Since $t^{\frac{1}{1-\lambda}}$ and $\Psi(t^{\frac{1}{\lambda}})$ are Young functions, Theorem 2.20 gives

$$\begin{aligned}\|f\|_{L^\Phi(\Omega;w)} &= \||f|^{1-\lambda}|f|^\lambda\|_{L^\Phi(\Omega;w)} \\ &\le C\|f\|^{1-\lambda}_{L^1(\Omega;w)}\|f\|^\lambda_{L^\Psi(\Omega;w)}.\end{aligned}$$

By Corollary 2.4 with exponents $p = \dfrac{1}{\lambda}$ and $q = \dfrac{1}{1-\lambda}$, we find for any $\varepsilon > 0$,

$$\|f\|_{L^\Phi(\Omega;w)} \le C\|f\|_{L^1(\Omega;w)} + \varepsilon\|f\|_{L^\Psi(\Omega;w)}.$$

□

Proposition 2.23. *Given two Young functions A, B, and a constant $c \ge 1$ such that*

$$B(t) \le cA(t)$$

then

$$\|f\|_{L^B(\Omega;w)} \le c\|f\|_{L^A(\Omega;w)}. \tag{2.13}$$

Proof. Define the sets

$$E = \{\lambda > 0 : \int_\Omega B\left(\frac{|f|}{\lambda}\right) wdx \leq 1\};$$
$$F = \{\mu > 0 : \int_\Omega A\left(\frac{|f|}{\mu}\right) wdx \leq 1\}.$$

Let $\mu' \in F$. Then, $\int_\Omega A\left(\frac{|f|}{\mu'}\right) wdx \leq 1$. Since $B(t) \leq cA(t)$ for some constant $c \geq 1$, then

$$\frac{1}{c}\int_\Omega B\left(\frac{|f|}{\mu'}\right) wdx \leq 1.$$

By the convexity of B,

$$\int_\Omega B\left(\frac{|f|}{c\mu'}\right) wdx \leq \frac{1}{c}\int_\Omega B\left(\frac{|f|}{\mu'}\right) wdx \leq 1.$$

That is, $c\mu' \in E$. Therefore, $cF \subset E$ and so

$$\|f\|_{L^B(\Omega;w)} = \inf(E) \leq \inf(cF) = c\inf(F) = c\|f\|_{L^A(\Omega;w)}.$$

□

Proposition 2.24. *Given any Young function Ψ and for any $S \subset \Omega$ with $w(S) > 0$, the norm of the indicator function $\mathbb{1}_S$ is*

$$\|\mathbb{1}_S\|_{L^\Psi(\Omega;w)} = \Psi^{-1}\left(w(S)^{-1}\right)^{-1}.$$

Proof. Set $\Lambda = \left\{\lambda > 0 : \int_S \Psi\left(\frac{1}{\lambda}\right) wdx \leq 1\right\}$. Notice, for $\lambda \in \Lambda$

$$w(S)\Psi\left(\frac{1}{\lambda}\right) = \int_\Omega \Psi\left(\frac{\mathbb{1}_S}{\lambda}\right) wdx \leq 1.$$

Since Ψ is strictly increasing and thus invertible, we find

$$\Psi^{-1}\left(w(S)^{-1}\right)^{-1} \leq \lambda \tag{2.14}$$

for all $\lambda \in \Lambda$. Set $\mu = \Psi^{-1}\left(w(S)^{-1}\right)^{-1}$ and consider

$$\begin{aligned}\int_S \Psi\left(\frac{1}{\Psi^{-1}\left(w(S)^{-1}\right)^{-1}}\right) w dx &= \int_S \Psi\left(\Psi^{-1}(w(S)^{-1})\right) w dx \\ &= \int_S w(S)^{-1} w dx \\ &= \frac{w(S)}{w(S)} \\ &= 1.\end{aligned}$$

Therefore $\mu \in \Lambda$ and by (2.14), we conclude

$$\|\mathbb{1}_S\|_{L^\Psi(\Omega;w)} = \Psi^{-1}\left(w(S)^{-1}\right)^{-1}.$$

□

2.3 Sobolev Spaces

The solution space to the partial differential equation we consider in this thesis is the degenerate Sobolev space $QH_0^1(\Omega; w)$. To understand the structure of this Sobolev space, it suffices to show what conditions must be satisfied to be a weak solution of the Dirichlet problem (4.1) (see definition 4.2) without the incorporation of lower-order terms. That is,

$$\begin{cases} -\frac{1}{w}\mathrm{Div}\left(Q\nabla u\right) &= f, \quad x \in \Omega; \\ u &= 0, \quad x \in \partial\Omega \end{cases} \tag{2.15}$$

where dependence on x is suppressed. Recall that the matrix-valued function Q satisfies (1.1) and consequently, $|Q(x)|_{\mathrm{op}} \in L^1_{loc}(\Omega)$. If u is a weak solution of (2.15), then

$$\int_\Omega \nabla u^T Q \nabla v \, dx = \int_\Omega f v \, w dx \tag{2.16}$$

holds for all "test functions" v where membership of v will be described momentarily. For (2.16) to hold, we require u and v to be functions that equal zero on the boundary and each integral in (2.16) to be finite. The minimum requirement to impose finiteness is

$$\sqrt{Q}\nabla u, \sqrt{Q}\nabla v \in L^2(\Omega); \;\; v, f \in L^2(\Omega; w).$$

Therefore, we need a collection of functions such that

$$\int_\Omega |v|^2 \, w dx + \int_\Omega |\sqrt{Q}\nabla v|^2 \, dx < \infty$$

for every v. The simplest functions to satisfy these conditions are Lipschitz functions with compact support in Ω. Indeed, since $|Q(x)|_{\text{op}} \in L^1_{loc}(\Omega)$ and Lipschitz functions are differentiable almost everywhere, $\nabla v \in L^\infty(E)$ where $E = supp(\nabla v)$. Therefore,

$$\int_\Omega |\sqrt{Q}\nabla v|^2 = \int_E |\sqrt{Q}\nabla v|^2 \, dx \leq \|\nabla v\|^2_{L^\infty(\Omega)} \int_E |Q|_{\text{op}} \, dx < \infty.$$

More, Lipschitz functions are continuous and so, v is bounded on its support S giving

$$\int_\Omega |v|^2 \, w dx = \int_S |v|^2 \, w dx \leq \|v^2\|_{L^\infty(\Omega;w)} w(S) < \infty.$$

Therefore, we complete the collection $Lip_0(\Omega)$ with respect to the norm

$$\|\varphi\|_{QH^1_0(\Omega;w)} = \|\varphi\|_{L^2(\Omega;w)} + \|\nabla\varphi\|_{\mathcal{L}^2(Q;\Omega)} \tag{2.17}$$

where $\|\nabla\varphi\|_{\mathcal{L}^2(Q;\Omega)} = \|\sqrt{Q}\nabla\varphi\|_{L^2(\Omega)}$. This collection is a set of equivalence classes of Cauchy sequences with the equivalence relation "$\sim$" defined by

$$\{f_n\} \sim \{g_n\} \iff \lim_{n\to\infty} \|f_n - g_n\|_{QH^1_0(\Omega;w)} = 0.$$

Note, given a Cauchy sequence $\{f_n\}$ of $Lip_0(\Omega)$ functions, we denote its equivalence class by $[\{f_n\}]$. By the definition of the $QH^1_0(\Omega; w)$ norm, both the sequences $\{f_n\}$ and $\{\nabla f_n\}$ are Cauchy in $L^2(\Omega; w)$ and $\mathcal{L}^2(Q; \Omega)$ respectively. Since $L^2(\Omega; w)$ and $\mathcal{L}^2(Q; \Omega)$ are complete spaces, (see [5]) there exist functions $u \in L^2(\Omega; w)$ and $\mathbf{g} \in \mathcal{L}^2(Q; \Omega)$ so that

$$\|f_n - u\|_{L^2(\Omega;w)} \to 0 \text{ as } n \to \infty$$

and

$$\|\nabla f_n - \mathbf{g}\|_{\mathcal{L}^2(Q;\Omega)} \to 0 \text{ as } n \to \infty.$$

Additionally, given any other sequence $\{h_n\} \in [\{f_n\}]$, we can write

$$\|h_n - u\|_{L^2(\Omega;w)} \leq \|f_n - u\|_{L^2(\Omega;w)} + \|f_n - h_n\|_{L^2(\Omega;w)}$$

and

$$\|\nabla h_n - \mathbf{g}\|_{\mathcal{L}^2(Q;\Omega)} \leq \|\nabla f_n - \mathbf{g}\|_{\mathcal{L}^2(Q;\Omega)} + \|\nabla f_n - \nabla h_n\|_{\mathcal{L}^2(Q;\Omega)}.$$

That is, the functions u and $\mathbf{g}$ are unique elements to the whole equivalence class $[\{f_n\}]$ in the space $L^2(\Omega;w) \times \mathcal{L}^2(Q;\Omega)$ equipped with the norm

$$\|(u,\mathbf{g})\|_{L^2(\Omega;w)\times\mathcal{L}^2(Q;\Omega)} = \|u\|_{L^2(\Omega;w)} + \|\mathbf{g}\|_{\mathcal{L}^2(Q;\Omega)}.$$

By the linearity and uniqueness of limits,

$$[c\{f_n\}] = c[\{f_n\}];$$
$$[\{f_n\}] + [\{h_n\}] = [\{f_n\} + \{h_n\}]$$

for any constant c. Consequently, we can define an injective, linear mapping $\mathcal{J} : QH_0^1(\Omega;w) \to L^2(\Omega;w) \times \mathcal{L}^2(Q;\Omega)$ by setting

$$\mathcal{J}([\{f_n\}]) = (u,\mathbf{g}).$$

Since $\|u\|_{L^2(\Omega;w)} = \lim_{n\to\infty} \|f_n\|_{L^2(\Omega;w)}$ and $\|\mathbf{g}\|_{\mathcal{L}^2(Q;\Omega)} = \lim_{n\to\infty} \|\nabla f_n\|_{\mathcal{L}^2(Q;\Omega)}$ by the reverse triangle inequality, the mapping $\mathcal{J}$ defines an isometric isomorphism from $QH_0^1(\Omega;w)$ to the closed subspace $\mathcal{J}(QH_0^1(\Omega;w))$. Since these spaces have the same algebraic structure, we do not make a distinction between them. Indeed, $QH_0^1(\Omega;w)$ has two equivalent perspectives. Specifically, $QH_0^1(\Omega;w)$ is a collection of equivalence classes of Cauchy sequences of $Lip_0(\Omega)$ functions and it is a collection of pairs $(u,\mathbf{g}) \in \mathcal{J}(QH_0^1(\Omega;w))$. One of the important properties of $QH_0^1(\Omega;w)$ is that it is a Hilbert space since the $QH_0^1(\Omega;w)$ norm is equivalent to the norm generated by inner product defined by

$$\langle f, g\rangle_{QH_0^1(\Omega;w)} = \int_\Omega f g\, w dx + \int_\Omega \nabla g^T Q \nabla f\, dx. \tag{2.18}$$

Indeed, there exists constants $c_1 = \frac{1}{2}, c_2 = 1$ such that

$$c_1 \|f\|^2_{QH_0^1(\Omega;w)} \leq \langle f, f\rangle_{QH_0^1(\Omega;w)} \leq c_2 \|f\|^2_{QH_0^1(\Omega;w)}$$

for all $f \in QH_0^1(\Omega;w)$. Additionally, Lemma 5.2 seen in Appendix A shows the $QH_0^1(\Omega;w)$ norm is equivalent to the $\mathcal{L}^2(Q;\Omega)$ norm under the assumption of Sobolev inequality (2.19).

In the literature, terminology is often abused by referring to elements in $QH_0^1(\Omega; w)$ as functions rather then pairs. Indeed, when we write $f \in QH_0^1(\Omega; w)$, we mean the pair $(f, \mathbf{h})$ where $f \in L^2(\Omega; w)$ and $\mathbf{h} \in \mathcal{L}^2(Q; \Omega)$ satisfying $\mathcal{J}([\{f_n\}]) = (f, \mathbf{h})$ for some $[\{f_n\}] \in QH_0^1(\Omega; w)$. We will denote $\mathbf{h}$ as ∇f however, we stress that it may not be the case that $\mathbf{h}$ is the weak derivative of f and may not even be uniquely determine by f. For instance, the authors in [6] construct an example where $f = 0$ almost everywhere, but $\nabla f = 1$ almost everywhere.

In this thesis, we will make two assumptions in relation to the function spaces $L^p(\Omega; w)$, $L^\Psi(\Omega; w)$ and the degenerate Sobolev space $QH_0^1(\Omega; w)$. First, we assume the projection mapping $\Pi : QH_0^1(\Omega; w) \to L^2(\Omega; w)$ defined by

$$\Pi((u, \nabla u)) = u$$

is a compact mapping. That is, the mapping Π satisfies the following definition.

Definition 2.25. *Let X and Y denote normed vector spaces. A mapping $T : X \to Y$ is a compact operator is one of the following conditions is satisfied:*

i) T maps bounded sets in X into relatively compact sets in Y.

ii) T maps bounded sequences in X into sequences in Y which contain convergent subsequences.

The proof that Π is a compact mapping relies on the assumption of a local Poincaré inequality along with additional background material that is beyond the scope of this study. However, the justification of this assumption is supported by the work in [7] and Theorem 7.22 in [8]. Secondly, we assume the existence of the following Sobolev inequalities in disjoint cases:

$$i)\ \|f\|_{L^2(\Omega;w)} \leq C\|\sqrt{Q}\nabla f\|_{L^2(\Omega)}; \tag{2.19}$$

$$ii)\ \|f\|_{L^\Psi(\Omega;w)} \leq C\|\sqrt{Q}\nabla f\|_{L^2(\Omega)}; \tag{2.20}$$

for all $f \in Lip_0(\Omega)$ where $C > 0$. These inequalities are adaptations of the Sobolev inequality

$$\left(\int_\Omega |f|^{2\sigma}\, w dx\right)^{\frac{1}{2\sigma}} \leq C\left(\int_\Omega |\sqrt{Q}\nabla f|^2\, dx\right)^{\frac{1}{2}}$$

studied extensively in [9], [10], where $\sigma \geq 1$ represents the gain factor. An interesting consequence of assuming the validity of a Sobolev inequality is how the roughness of the coefficients is dependent on the dual of σ. That is, assuming (2.19) holds, the coefficient functions $\mathbf{H}$ and F from (4.1) are members of $L^{\infty}(\Omega; w)$ since $\sigma = 1$ and its dual $\sigma' = \infty$. Note, this is the roughest geometry and consequently restricts the membership of $\mathbf{H}$ and F more than when $\sigma > 1$. If we assume (2.20) holds, then $\sigma > 1$ is not quantifiable but we do see the gain factor $\log(e - 1 + t)^q$ and so, we may let $\mathbf{H} \in L^{\Psi_{\mathrm{H}}}(\Omega; w)$ and $F \in L^{\Psi_F}(\Omega; w)$ where $\Psi_{\mathrm{H}} = e^{t^{\frac{2}{\gamma}}} - 1$ and $\Psi_F = e^{t^{\frac{1}{\gamma}}} - 1$ for $0 < \gamma \leq \frac{q}{2}$, each related to the dual of the log-gain forcing exponential integrability. Note, since $t^2 \leq ct^2 \log(e - 1 + t)^q$ for all $t \geq 0$ when $c = \frac{1}{\log(e-1)^q}$, Proposition 2.23 gives

$$\|f\|_{L^2(\Omega;w)} \leq c\|f\|_{L^{\Psi}(\Omega;w)}.$$

Therefore, if (2.20) holds, then $\|f\|_{L^2(\Omega;w)} \leq \tilde{C}\|\sqrt{Q}\nabla f\|_{L^2(\Omega)}$. The Sobolev inequalities (2.19) and (2.20) can be extended to hold for all functions in $QH_0^1(\Omega; w)$ through the following density arguments.

Lemma 2.26. *Suppose that the Sobolev inequality* (2.19) *holds for all* $f \in Lip_0(\Omega)$. *Then,* (2.19) *holds for all* $u \in QH_0^1(\Omega; w)$.

Proof. Let $u \in QH_0^1(\Omega; w)$. Then, there is a representative sequence $\{f_n\} \subset Lip_0(\Omega)$ such that

$$\|f_n - u\|_{L^2(\Omega;w)} \to 0 \text{ as } n \to \infty$$

and

$$\|\sqrt{Q}\nabla f_n - \sqrt{Q}\nabla u\|_{L^2(\Omega)} \to 0 \text{ as } n \to \infty.$$

Fix $n \in \mathbb{N}$. Then, by the triangle inequality and Sobolev inequality (2.19)

$$\begin{aligned}\|u\|_{L^2(\Omega;w)} &\leq \|f_n - u\|_{L^2(\Omega;w)} + \|f_n\|_{L^2(\Omega;w)} \\ &\leq \|f_n - u\|_{L^2(\Omega;w)} + \|\sqrt{Q}\nabla f_n\|_{L^2(\Omega)}.\end{aligned}$$

Since $\|\sqrt{Q}\nabla u\|_{L^2(\Omega)} = \lim_{n\to\infty} \|\sqrt{Q}\nabla f_n\|_{L^2(\Omega)}$, sending $n \to \infty$ gives

$$\|u\|_{L^2(\Omega;w)} \leq C\|\sqrt{Q}\nabla u\|_{L^2(\Omega)}.$$

□

Lemma 2.27. *Suppose that the Sobolev inequality* (2.20) *holds for all* $f \in Lip_0(\Omega)$. *Then,* (2.20) *holds for all* $u \in QH_0^1(\Omega; w)$.

Proof. Let $u \in QH_0^1(\Omega; w)$. Then, there is a sequence $\{f_n\}$ of Lipschitz functions with compact support on Ω such that

$$\|f_n - u\|_{L^2(\Omega;w)} \to 0 \text{ as } n \to \infty$$

and

$$\|\sqrt{Q}\nabla f_n - \sqrt{Q}\nabla u\|_{L^2(\Omega)} \to 0 \text{ as } n \to \infty.$$

By Sobolev inequality (2.20), for $n, m \in \mathbb{N}$,

$$\|f_n - f_m\|_{L^\Psi(\Omega;w)} \leq \|\sqrt{Q}\nabla f_n - \sqrt{Q}\nabla f_m\|_{L^2(\Omega)} \leq \|f_n - f_m\|_{QH_0^1(\Omega;w)}$$

and so, $\{f_n\}$ is Cauchy in $L^\Psi(\Omega; w)$. Since $L^\Psi(\Omega; w)$ and $L^2(\Omega; w)$ are complete, as shown in [3], [5], and $L^\Psi(\Omega; w) \subset L^2(\Omega; w)$, it follows that

$$\|f_n - u\|_{L^\Psi(\Omega;w)} \to 0 \text{ as } n \to \infty.$$

Therefore, by following the same argument in Lemma 2.27, we have that

$$\|u\|_{L^\Psi(\Omega;w)} \leq C\|\sqrt{Q}\nabla u\|_{L^2(\Omega)}.$$

□

3 Functional Analysis

The degenerate Sobolev space $QH_0^1(\Omega; w)$ is a Hilbert space. Consequently, the existence of weak solutions can be studied using results from functional analysis such as the Lax-Milgram Theorem and the Fredholm Alternative. These theorems are well-documented in the literature; however, we provide their proofs for completeness, as they are fundamental to our work in Chapter 4. The proofs in this chapter are adapted from Chapter 5 of [8] with additional details included for clarity. All supporting lemmas and theorems referenced in the main proofs are provided in Appendix B.

3.1 Banach and Hilbert Spaces

Definition 3.1. *A Banach space $\mathscr{D}$ is a normed vector space that is complete with respect to a norm $\|\cdot\|_{\mathscr{D}}$.*

Definition 3.2. *A Hilbert space $\mathscr{H}$ is a Banach space with the norm $\|\cdot\|_{\mathscr{H}}$ provided with an inner product $\langle\cdot,\cdot\rangle : \mathscr{H} \to \mathscr{H}$ compatible with its norm. That is, for any element $x \in \mathscr{H}$, $\|x\|_{\mathscr{H}} = \langle x, x\rangle^{\frac{1}{2}}$.*

Definition 3.3. *Let V be a normed linear space. A functional $\mathscr{F}$ on V is a mapping from V into $\mathbb{R}$. A functional $\mathscr{F}$ is linear if for all $x, y \in V$ $\alpha \in \mathbb{R}$,*

$$\mathscr{F}(x+y) = \mathscr{F}(x) + \mathscr{F}(y)$$
$$\mathscr{F}(\alpha x) = \alpha \mathscr{F}(x)$$

and is bounded if there exists a $C > 0$ such that

$$|\mathscr{F}(x)| \leq C\|x\|$$

for all $x \in V$. The space of all bounded linear functionals is called the dual space of V and is denoted V^*.

Definition 3.4. *Let V be a vector space over $\mathbb{R}$. A bilinear form $\mathscr{B}$ on V is a function of two vectors such that $\mathscr{B} : V \times V \to \mathbb{R}$ satisfies linearity in the first and second argument. Specifically, for $u, v, z \in V, \lambda \in \mathbb{R}$,*

$$\begin{aligned}
\mathscr{B}(u+v, z) &= \mathscr{B}(u, z) + \mathscr{B}(v, z); \\
\mathscr{B}(\lambda u, v) &= \lambda \mathscr{B}(u, v); \\
\mathscr{B}(u, v+z) &= \mathscr{B}(u, v) + \mathscr{B}(u, z); \\
\mathscr{B}(u, \lambda v) &= \lambda \mathscr{B}(u, v).
\end{aligned}$$

3.2 The Lax-Milgram Theorem

Theorem 3.5. *Let $\mathscr{H}$ be a Hilbert space and assume $\mathscr{B} : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ is a bilinear mapping satisfying*

i) $\mathscr{B}$ is bounded on $\mathscr{H}$. That is, there is a $c > 0$ such that for all $x, y \in \mathscr{H}$ then

$$|\mathscr{B}(x, y)| \leq c\|x\|\|y\|.$$

ii) $\mathscr{B}$ is coercive on $\mathscr{H}$. That is, there is a $C > 0$ such that for all $x \in \mathscr{H}$ we have

$$\mathscr{B}(x, x) \geq C\|x\|^2.$$

Then, given $\mathscr{F} \in \mathscr{H}^*$, there is a unique $u \in \mathscr{H}$ so that $\mathscr{F}(x) = \mathscr{B}(x, u)$ for all $x \in \mathscr{H}$.

Proof. For each fixed $y \in \mathscr{H}$, $\mathscr{B}(x, y)$ is a bounded linear functional on $\mathscr{H}$. The Riesz-Representation Theorem, Theorem 6.2 in Appendix B, asserts the existence of a unique element $w \in \mathscr{H}$ satisfying

$$\mathscr{B}(x, y) = \langle x, w\rangle \tag{3.1}$$

for every $x, y \in \mathscr{H}$. Define $T(y) = w$ whenever (3.1) holds. Since w is uniquely determined for each $y \in \mathscr{H}$, T is well-defined. So,

$$\mathscr{B}(x, y) = \langle x, T(y)\rangle \tag{3.2}$$

holds for all $x, y \in \mathscr{H}$. Let $\lambda_1, \lambda_2 \in \mathbb{R}, x_1, x_2 \in \mathscr{H}$ then by (3.1), (3.2), and linearity of $\mathscr{B}$, for any $x \in \mathscr{H}$,

$$\begin{aligned}\langle x, T(\lambda_1 x_1 + \lambda_2 x_2)\rangle &= \mathscr{B}(x, \lambda_1 x_1 + \lambda_2 x_2)\\ &= \mathscr{B}(x, \lambda_1 x_1) + \mathscr{B}(x, \lambda_2 x_2)\\ &= \lambda_1\langle x, T(x_1)\rangle + \lambda_2\langle x, T(x_2)\rangle\\ &= \langle x, \lambda_1 T(x_1) + \lambda_2 T(x_2)\rangle.\end{aligned}$$

Therefore, $T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$ and so, T is linear. Notice, if $y = 0$ then

$$\|T(y)\| \leq c\|y\|$$

for any $c > 0$. If $y \neq 0$ then

$$\|T(y)\|^2 = \langle T(y), T(y)\rangle = \mathscr{B}(T(y), y) \leq c\|T(y)\|\|y\|$$

by hypothesis (i). Therefore, T is bounded. More, by hypothesis (ii),

$$C\|y\|^2 \leq \mathscr{B}(y, y) = \langle y, T(y)\rangle \leq \|T(y)\|\|y\|$$

for each $y \in \mathscr{H}$. Consequently, T is injective, has closed range by Lemma 6.3, and T^{-1}, whose existence will be asserted momentarily, is bounded . Let $\mathscr{R}(T)$ be the range of T and suppose $\mathscr{R}(T) \neq \mathscr{H}$. Then, by the Projection Theorem, Theorem 6.4 in Appendix B, there exists a non-zero element $z \in \mathscr{H}$ such that $z \in \mathscr{R}(T)^{\perp}$. But, by hypothesis (ii),

$$C\|z\|^2 \leq \mathscr{B}(z, z) = \langle z, T(z)\rangle = 0$$

since $T(z) \in \mathscr{R}(T)$, a contradiction. Therefore, T is surjective. Hence, T^{-1} exists and is linear and continuous by Theorem 6.1. Let $\mathscr{F} \in \mathscr{H}^*$. Then, by Theorem 6.2, there exists a unique $v \in \mathscr{H}$ so that

$$\mathscr{F}(x) = \langle x, v\rangle.$$

Set $u = T^{-1}(v)$ then $\mathscr{B}(x, u) = \langle x, v\rangle = \mathscr{F}(x)$. That is,

$$\mathscr{F}(x) = \mathscr{B}(x, u)$$

for all $x \in \mathscr{H}$.

□

3.3 The Fredholm Alternative

Theorem 3.6. *Let T be a compact linear mapping of a normed vector space V into itself. Then either (i) the homogeneous equation*

$$x - T(x) = 0$$

has a nontrivial solution $x \in V$ or (ii) for each $y \in V$, the equation

$$x - T(x) = y$$

has a uniquely determined solution $x \in V$. Furthermore, in case (ii), the operator $(I - T)^{-1}$ whose existence is asserted there is also bounded.

Proof. Let $S = I - T$ where I is the identity mapping and $T : V \to V$ is a compact linear mapping. Since I is continuous and T is compact, S is a continuous linear operator. Define $\mathscr{N} = S^{-1}(0) = \{x \in V : S(x) = 0\}$ as the null space of S. Note, by linearity and continuity of S, and the uniqueness of limits, the null space is a closed subspace of V. To prove the Fredholm alternative, it suffices to prove the following four lemmas.

Lemma 3.7. *There exists a constant C such that*

$$dist(x, \mathscr{N}) \leq C\|S(x)\| \tag{3.3}$$

for all $x \in V$ where $dist(x, \mathscr{N}) = \inf_{y \in \mathscr{N}} \|x - y\|$.

Proof. Suppose the negation is true. That is, for every constant C, there exists an $x \in V$ such that

$$dist(x, \mathscr{N}) > C\|S(x)\|.$$

Let $C = n$ where $n \in \mathbb{N}$. Then, construct a sequence $\{\tilde{x}_n\}_{n \in \mathbb{N}}$ where

$$dist(\tilde{x}_n, \mathscr{N}) > C\|S(\tilde{x}_n)\|$$

holds for each n. Set $x_n = \dfrac{\tilde{x}_n}{\|S(\tilde{x}_n)\|}$ where $\|S(\tilde{x}_n)\| \neq 0$ otherwise $\tilde{x}_n \in \mathscr{N}$ and $dist(\tilde{x}_n, \mathscr{N}) = 0$. Then, $\|S(x_n)\| = 1$ and notice,

$$\begin{aligned}
d(x_n, \mathscr{N}) &= \inf_{y \in \mathscr{N}} \|x_n - y\| \\
&= \inf_{y \in \mathscr{N}} \left\| \frac{\tilde{x}_n}{\|S(\tilde{x}_n)\|} - y \right\| \\
&= \inf_{y \in \mathscr{N}} \left\| \frac{\tilde{x}_n - \|S(\tilde{x}_n)\| y}{\|S(\tilde{x}_n)\|} \right\| \\
&\geq \frac{1}{\|S(\tilde{x}_n)\|} \inf_{y' \in \mathscr{N}} \|\tilde{x}_n - y'\| \\
&> n \\
&= n\|S(x_n)\|
\end{aligned}$$

since $\|S(\tilde{x}_n)\| > 0$ and $\mathscr{N}$ is a closed subspace of V. Consequently, $dist(x_n, \mathscr{N}) \to \infty$ as $n \to \infty$. Let $\varepsilon > 0$ be given. Then, for some $\tilde{y} \in \mathscr{N}$,

$$\|x_n - \tilde{y}\| \leq dist(x_n, \mathscr{N}) + \varepsilon.$$

Let $\varepsilon = dist(x_n, \mathscr{N})$ and construct a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathscr{N}$ such that

$$\|x_n - y_n\| \leq 2dist(x_n, \mathscr{N})$$

holds for each n. By definition of the infimum, $dist(x_n, \mathscr{N}) \leq \|x_n - y_n\|$. Write $z_n = \dfrac{x_n - y_n}{\|x_n - y_n\|}$ where $\|x_n - y_n\| \neq 0$ since $dist(x_n, \mathscr{N}) > 0$. Then, $\|z_n\| = 1$ and

$$\|S(z_n)\| = \frac{\|S(x_n) - S(y_n)\|}{\|x_n - y_n\|} = \frac{1}{\|x_n - y_n\|} \leq \frac{1}{dist(x_n, \mathscr{N})}.$$

Sending $n \to \infty$ shows $\|S(z_n)\| \to 0$ and so, $S(z_n) \to 0$ in V. Since $\{z_n\}_{n\in\mathbb{N}}$ is bounded and T is compact, $\{T(z_n)\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{T(z_{n_k})\}_{k\in\mathbb{N}}$ converging to some $y_0 \in V$. Notice, $z_{n_k} = S(z_{n_k}) + T(z_{n_k})$ for all $k \in \mathbb{N}$. Sending $k \to \infty$ shows $z_{n_k} \to y_0$ since $S(z_{n_k}) \to 0$ as every subsequence of a convergent sequence converges to the same limit. Consequently, $y_0 \in \mathscr{N}$ by the continuity of S and uniqueness of limits. That is, $\lim\limits_{k\to\infty} S(z_{n_k}) = 0$ but $\lim\limits_{k\to\infty} S(z_{n_k}) = S(y_0)$. This leads to a contradiction as

$$\begin{aligned}
dist(z_n, \mathscr{N}) &= \inf_{y\in\mathscr{N}} \|z_n - y\| \\
&= \inf_{y\in\mathscr{N}} \left\| \frac{x_n - y_n}{\|x_n - y_n\|} - y \right\| \\
&= \inf_{y\in\mathscr{N}} \left\| \frac{x_n - y_n - \|x_n - y_n\| y}{\|x_n - y_n\|} \right\| \\
&\geq \frac{1}{\|x_n - y_n\|} dist(x_n, \mathscr{N}) \\
&\geq \frac{1}{2}
\end{aligned}$$

for all $n \in \mathbb{N}$. So, $\|z_{n_k} - y_0\| \geq \frac{1}{2}$ for all $k \in \mathbb{N}$ contradicting $\|z_{n_k} - y_0\| \to 0$ as $k \to \infty$.

□

Lemma 3.8. *Let $\mathscr{R} = S(V)$ be the range of S. Then, $\mathscr{R}$ is a closed subspace of V.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in V whose image $\{S(x_n)\}_{n\in\mathbb{N}}$ converges to an element $y \in V$. To show $\mathscr{R}$ is closed, there must exist an $x \in V$ such that

$y = S(x)$. By Lemma 3.7, the sequence $\{dist(x_n, \mathscr{N})\}_{n\in\mathbb{N}}$ is bounded since $\{S(x_n)\}_{n\in\mathbb{N}}$ is convergent and so $\{\|S(x_n)\|\}_{n\in\mathbb{N}}$ is convergent by the reverse triangle inequality and therefore bounded. Construct the sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathscr{N}$ as done in Lemma 3.7 to obtain

$$dist(x_n, \mathscr{N}) \leq \|x_n - y_n\| \leq 2dist(x_n, \mathscr{N})$$

for each n. Write $w_n = x_n - y_n$ then $\{\|w_n\|\}_{n\in\mathbb{N}}$ is bounded and $S(w_n) \to y$ in V as $n \to \infty$. Since T is compact, $\{T(w_{n_k})\}_{k\in\mathbb{N}}$ converges to some $w_0 \in V$. Since $w_{n_k} = S(w_{n_k}) + T(w_{n_k})$ for each k, $w_{n_k} \to y + w_0$ as $k \to \infty$. By the continuity of S and uniqueness of limits,

$$S(y + w_0) = \lim_{k\to\infty} S(w_{n_k}) = y.$$

Letting $x = y + w_0$ concludes the proof.

□

Lemma 3.9. *If $\mathscr{N} = \{0\}$, then $\mathscr{R} = V$. That is, if case (i) of Theorem 3.6 does not hold, then case (ii) is true.*

Proof. By Lemma 3.8, the sets $\mathscr{R}_j$ defined by $S^j(V)$ for $j \in \mathbb{N}$ form a non-increasing sequence of closed subspaces of V. Suppose for all $j \in \mathbb{N}$ that $\mathscr{R}_{j+1}$ is a proper closed subspace of $\mathscr{R}_j$. Then, by Lemma 6.5, there is a sequence $\{y_n\}_{n\in\mathbb{N}} \subset V$ such that $y_n \in \mathscr{R}_n$, $\|y_n\| = 1$, and $dist(y_n, \mathscr{R}_{n+1}) \geq \frac{1}{2}$ for each n. Then if $n > m$,

$$T(y_m) - T(y_n) = y_m - (y_n + S(y_m) - S(y_n))$$

where $y_n \in \mathscr{R}_n \subset \mathscr{R}_{m+1}$, $S(y_n) \in \mathscr{R}_{n+1} \subset \mathscr{R}_{m+1}$, and $S(y_m) \in \mathscr{R}_{m+1}$. Setting $y' = y_n + S(y_m) - S(y_n)) \in \mathscr{R}_{m+1}$ it follows that

$$\begin{aligned} \|T(y_m) - T(y_n)\| &= \|y_m - y'\| \\ &\geq dist(y_m, \mathscr{R}_{m+1}) \\ &\geq \frac{1}{2}. \end{aligned}$$

Since this is true for all n, m, there is no subsequence of $\{T(y_n)\}_{n\in\mathbb{N}}$ that is Cauchy and therefore can not converge contradicting the compactness of T since $\{y_n\}_{n\in\mathbb{N}}$ is bounded. Consequently, there exists a $k \in \mathbb{Z}$ such that $\mathscr{R}_j = \mathscr{R}_k$ for all $j \geq k$. Otherwise, by omitting the indices where they are equal, we would obtain a decreasing sequence of closed subspaces. Let y be an arbitrary element of V. Then, $S^k(y) \in \mathscr{R}_k = \mathscr{R}_{k+1}$ and so, there exists an $x \in V$ such that $S^{k+1}(x) = S^k(y)$. Notice,

$$S^k(y) - S^{k+1}(x) = S^k(y - S(x)) = 0.$$

Since $\mathscr{N} = \{0\}$, then $S^{-k}(0) = S^{-1}(0)$ therefore $y = S(x)$ proving $\mathscr{R} = V$.

□

Lemma 3.10. *If $\mathscr{R} = V$, then $\mathscr{N} = \{0\}$. Consequently either case (i) or case (ii) holds in Theorem 3.6.*

Proof. Define a non-decreasing sequence of closed subspaces $\{\mathscr{N}_j\}_{j\in\mathbb{N}}$ by setting $\mathscr{N}_j = S^{-j}(0)$. This is justified since for any $z \in \mathscr{N}_j = \{x \in V : S^j(x) = 0\}$, $S^{j+1}(z) = S(S^j(z)) = S(0) = 0$ and so, $z \in \mathscr{N}_{j+1}$. Suppose for all $j \in \mathbb{N}$ that $\mathscr{N}_j$ is a proper closed subspace of $\mathscr{N}_{j+1}$. By Lemma 6.5, we construct the sequence $\{y_j\}_{j\in\mathbb{N}}$ such that $y_j \in \mathscr{N}_{j+1}$, $\|y_j\| = 1$, and $dist(y_j, \mathscr{N}_j) \geq \frac{1}{2}$ for each j. Then, if $k > j$

$$T(y_k) - T(y_j) = y_k - (y_j + S(y_k) - S(y_j))$$

where $y_j \in \mathscr{N}_{j+1} \subset \mathscr{N}_k$, $S^k(S(y_k)) = S^{k+1}(y_k) = 0$ so $S(y_k) \in \mathscr{N}_k$ and $S^k(S(y_j)) = S^{k+1}(y_j) = S^m(S^{j+1}(y_j)) = S^m(0) = 0$ giving $S(y_j) \in \mathscr{N}_k$. Set $y' = y_j + S(y_k) - S(y_j) \in \mathscr{N}_k$ then

$$\begin{aligned} \|T(y_k) - T(y_j)\| &= \|y_k - y'\| \\ &\geq dist(y_k, \mathscr{N}_k) \\ &\geq \frac{1}{2}, \end{aligned}$$

contradicting the compactness of T. Consequently, there exists an $l \in \mathbb{Z}$ such that $\mathscr{N}_j = \mathscr{N}_l$ for all $j \geq l$. Let $y \in \mathscr{N}_l$ and since $S^l(V) = \mathscr{R} = V$, there exists an $x \in V$ such that $y = S^l(x)$. Notice, $S^l(y) = S^{2l}(x) = 0$ and so, $x \in \mathscr{N}_{2l} = \mathscr{N}_l$. Hence, $y = S^l(x) = 0$. Since $y \in \mathscr{N}_l$ was arbitrary, $\mathscr{N}_l = \{0\}$. Therefore, $\mathscr{N} = \{0\}$ since $\{0\} \subset \mathscr{N} \subset \mathscr{N}_l = \{0\}$.

□

To finish the proof, all that is left to show is boundedness of S^{-1} in case (ii) whose existence is asserted since S is bijective. By Lemma 3.7, there exists a constant C so that

$$dist(\tilde{x}, \mathscr{N}) \leq C\|S(\tilde{x})\|$$

holds for all $\tilde{x} \in V$. Since $\mathscr{N} = \{0\}$, it follows that $dist(\tilde{x}, \mathscr{N}) = \|\tilde{x}\|$. Setting $\tilde{x} = S^{-1}(x)$ then,

$$\|S^{-1}(x)\| \leq C\|x\|$$

for all $x \in V$. The Fredholm Alternative is therefore completely proved.

□

4 Degenerate Elliptic Equations

4.1 The Dirichlet Problem

The Dirichlet problem associated to this study is as follows, where the degenerate matrix valued function Q and non-negative weight $w \in L^1_{loc}(\Omega)$ satisfy (1.1), $f \in L^2(\Omega)$, and we suppress dependence on $x \in \Omega$:

$$\begin{cases} -\frac{1}{w}\text{Div}\,(Q\nabla u) + \mathbf{HT}u + Fu &= f, \quad x \in \Omega; \\ u &= 0, \quad x \in \partial\Omega. \end{cases} \tag{4.1}$$

In (4.1), $F : \Omega \to \mathbb{R}$ is a non-negative real-valued function, and for $N \in \mathbb{N}$, $\mathbf{H} : \Omega \to \mathbb{R}^N$ is a vector-valued function of the form $\mathbf{H}(x) = (h_1(x), ..., h_N(x))$. It is unnecessary to assume continuity or boundedness of the coefficient functions $\mathbf{H}$ and F in general. The only requirement is that these functions belong to various Orlicz and Lebesgue function spaces depending on the form of the Sobolev inequality one is given. $\mathbf{T}$ is a vector of first order vector fields assumed to be subunit with respect to the quadratic form $\xi^T Q\xi$ where $\xi \in \mathbb{R}^n$. Specifically,

$$\mathbf{T}(x) = (T_1(x), T_2(x), ..., T_N(x))$$

where the first order vector field $T_i(x) = \mathbf{t}_i(x) \cdot \nabla = \sum_{j=1}^{n} t^i_j(x)\frac{\partial}{\partial x_j}$ is identified with the vector function $\mathbf{t}_i(x) = (t^i_1(x), ..., t^i_n(x))$ that is subunit with respect to the quadratic form $\xi^T Q\xi$. That is, each t^i_j is defined and measurable in Ω for each $1 \leq j \leq n$ and

$$|\mathbf{t}_i \cdot \xi|^2 \leq \frac{|\sqrt{Q}\,\xi\,|^2}{w} \tag{4.2}$$

for all $\xi \in \mathbb{R}^n$, and almost every $x \in \Omega$. $\mathbf{HT}$ denotes the standard dot product of $\mathbf{H}$ with $\mathbf{T}$ and given a real valued function $u(x)$, the term $\mathbf{HT}u$ denotes

$$\mathbf{HT}u = \sum_{i=1}^{N} h_i(x)T_i(x)u(x) = \sum_{i=1}^{N}\sum_{j=1}^{n} h_i(x)t^i_j(x)\frac{\partial}{\partial x_j}u(x).$$

Note, (4.2) allows us to establish the following pointwise estimate.

Lemma 4.1. *Fix $N \in \mathbb{N}$ and $\mathbf{T}(x) = (T_1(x), ..., T_N(x))$ where each $T_i(x) = \sum_{j=1}^{n} t_j^i(x) \frac{\partial}{\partial x_j}$ is identified with the vector $\mathbf{t}_i(x) = (t_1^i, ..., t_n^i)$ that satisfies*

$$|\mathbf{t}_i(x) \cdot \xi|^2 \leq \frac{|\sqrt{Q(x)}\,\xi|^2}{w(x)}$$

for every $\xi \in \mathbb{R}^n$, and almost every $x \in \Omega$. Then, for any $(u, \nabla u) \in QH_0^1(\Omega; w)$, we have the pointwise estimate

$$|\mathbf{T}(x)u(x)| \leq \frac{\sqrt{N}}{\sqrt{w(x)}} |\sqrt{Q(x)}\nabla u(x)|. \tag{4.3}$$

Proof. Fix $(u, \nabla u) \in QH_0^1(\Omega; w)$ and consider the following estimate:

$$\begin{aligned}
|\mathbf{T}(x)u(x)| &= \left(\sum_{i=1}^{N} (T_i(x)u(x))^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{N} (\mathbf{t}_i(x) \cdot \nabla u(x))^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^{N} \frac{|\sqrt{Q(x)}\nabla u(x)|^2}{w(x)} \right)^{\frac{1}{2}} \\
&= \frac{\sqrt{N}}{\sqrt{w(x)}} |\sqrt{Q(x)}\nabla u(x)|. \quad \square
\end{aligned}$$

Definition 4.2. *A weak solution $(u, \nabla u) \in QH_0^1(\Omega; w)$ of the Dirichlet problem (4.1) exists if the following expression*

$$\int_\Omega \nabla u^T Q \nabla v \, dx + \int_\Omega v \mathbf{H}\mathbf{T} u \, w dx + \int_\Omega v F u \, w dx = \int_\Omega f v \, w dx \tag{4.4}$$

holds for any Lipschitz function v with compact support in Ω.

We will always refer to these Lipschitz functions as "test functions" for our weak solution $(u, \nabla u) \in QH_0^1(\Omega; w)$. Before proceeding, we establish the left-hand side of (4.4) is a well-defined bilinear form acting on the Hilbert space $QH_0^1(\Omega; w) \times QH_0^1(\Omega; w)$ to use the theory of Chapter 3.

Lemma 4.3. *$\mathscr{L}(u,v) = \int_\Omega \nabla u^T Q \nabla v \, dx + \int_\Omega v\mathbf{H}\mathbf{T}u \, w dx + \int_\Omega vFu \, w dx$ is well defined on $QH_0^1(\Omega;w) \times QH_0^1(\Omega;w)$ in the following two cases:*

i) If Sobolev inequality (2.19) holds and $\mathbf{H}, F \in L^\infty(\Omega;w)$. Note, $\mathbf{H} \in L^\infty(\Omega;w)$ means for $\mathbf{H}(x) = (h_1(x), ..., h_N(x))$, $h_i(x) \in L^\infty(\Omega;w)$ for each $1 \le i \le n$.

ii) If Sobolev inequality (2.20) holds and $\mathbf{H} \in L^{\Psi_H}(\Omega;w)$, $F \in L^{\Psi_F}(\Omega;w)$, where $\Psi_H = e^{t^{\frac{2}{\gamma}}} - 1$ and $\Psi_F = e^{t^{\frac{1}{\gamma}}} - 1$ for $0 < \gamma \le \frac{q}{2}$.

Proof. Item *i*) and *ii*) are proved in Lemma 4.5 and Lemma 4.12, respectively. □

Lemma 4.4. *$\mathscr{L}(u,v) = \int_\Omega \nabla u^T Q \nabla v \, dx + \int_\Omega v\mathbf{H}\mathbf{T}u \, w dx + \int_\Omega vFu \, w dx$ is a bilinear form acting on $QH_0^1(\Omega;w) \times QH_0^1(\Omega;w)$.*

Proof. For $\mathscr{L}$ to be a bilinear form acting on $QH_0^1(\Omega;w) \times QH_0^1(\Omega;w)$, linearity in the first and second argument must be satisfied. Let $u, v, z \in QH_0^1(\Omega;w)$ and $\lambda \in \mathbb{R}$. Then,

$$\mathscr{L}(u+z,v) = \int_\Omega \nabla(u+z)^T Q \nabla v \, dx + \int_\Omega v\mathbf{H}\mathbf{T}(u+z) \, w dx + \int_\Omega vF(u+z) \, w dx$$

and since,

$$\int_\Omega \nabla(u+z)^T Q \nabla v \, dx = \int_\Omega \nabla u^T Q \nabla v \, dx + \int_\Omega \nabla z^T Q \nabla v \, dx$$

by the linearity of ∇, we find,

$$\mathscr{L}(u+z,v) = \mathscr{L}(u,v) + \mathscr{L}(z,v).$$

Next, we consider

$$\begin{aligned}
\mathscr{L}(\lambda u, v) &= \int_\Omega \nabla \lambda u^T Q \nabla v \, dx + \int_\Omega v\mathbf{H}\mathbf{T}(\lambda u) \, w dx + \int_\Omega vF(\lambda u) \, w dx \\
&= \lambda \int_\Omega \nabla u^T Q \nabla v dx + \lambda \int_\Omega v\mathbf{H}\mathbf{T}u \, w dx + \lambda \int_\Omega vFu \, w dx \\
&= \lambda \mathscr{L}(u,v).
\end{aligned}$$

That is, $\mathscr{L}$ is linear in the first argument. Note, an identical argument shows $\mathscr{L}$ is indeed linear in the second argument. Since, $u, v, z \in QH_0^1(\Omega;w)$ and $\lambda \in \mathbb{R}$ were arbitrary, $\mathscr{L}$ is a bilinear form acting on $QH_0^1(\Omega;w) \times QH_0^1(\Omega;w)$. □

4.2 Existence of Solutions

4.2.1 Bounded Coefficients Case

Suppose Sobolev inequality (2.19) holds. Let the coefficient functions $\mathbf{H}, F \in L^\infty(\Omega; w)$ with

$$\|\mathbf{H}\|_{L^\infty(\Omega;w)} < \frac{1}{C\sqrt{N}} \tag{4.5}$$

where $C > 0$ is from Sobolev inequality (2.19). Notice,

$$\mathscr{F}(v) = \int_\Omega fv\, w dx$$

is a bounded linear functional on $QH_0^1(\Omega; w)$ for a fixed f in $L^2(\Omega)$ and so, we proceed showing boundedness and almost coercivity of $\mathscr{L}$.

Lemma 4.5. *There exists a constant $C > 0$ such that*

$$|\mathscr{L}(u, v)| \leq C\|u\|_{QH_0^1(\Omega;w)}\|v\|_{QH_0^1(\Omega;w)}$$

for all $u, v \in QH_0^1(\Omega; w)$.

Proof. Let $u, v \in QH_0^1(\Omega; w)$. Then, by Theorem 2.5

$$\begin{aligned}
|\mathscr{L}(u, v)| &\leq \int_\Omega |\sqrt{Q}\nabla u \sqrt{Q}\nabla v|\, dx + \int_\Omega |v\mathbf{H}\mathbf{T}u|\, w dx + \int_\Omega |vFu|\, w dx \\
&\leq \|\sqrt{Q}\nabla u\|_{L^2(\Omega)}\|\sqrt{Q}\nabla v\|_{L^2(\Omega)} + \int_\Omega |v||\mathbf{H}||\mathbf{T}u|\, w dx + \int_\Omega |v||F||u|\, w dx \\
&\leq \|u\|_{QH_0^1(\Omega;w)}\|v\|_{QH_0^1(\Omega;w)} + I + II.
\end{aligned}$$

We estimate terms I and II separately beginning with the former. By Lemma 4.1 and Theorem 2.6, we have that

$$\begin{aligned}
I &= \int_\Omega |v||\mathbf{H}||\mathbf{T}u|\, w dx \\
&\leq \sqrt{N}\int_\Omega |v\sqrt{w}||\mathbf{H}||\sqrt{Q}\nabla u|\, dx \\
&\leq \sqrt{N}\|\mathbf{H}\|_{L^\infty(\Omega;w)}\, \|v\sqrt{w}\|_{L^2(\Omega)}\, \|\sqrt{Q}\nabla u\|_{L^2(\Omega)} \\
&\leq \sqrt{N}\|\mathbf{H}\|_{L^\infty(\Omega;w)}\|u\|_{QH_0^1(\Omega;w)}\|v\|_{QH_0^1(\Omega;w)},
\end{aligned}$$

where $\|v\sqrt{w}\|_{L^2(\Omega)} = \|v\|_{L^2(\Omega;w)}$. Transitioning to II, Theorem 2.6 gives

$$\begin{aligned} II &= \int_\Omega |v||F||u|\, w dx \\ &\leq \|F\|_{L^\infty(\Omega;w)} \|v\sqrt{w}\|_{L^2(\Omega)} \|u\sqrt{w}\|_{L^2(\Omega)} \\ &\leq \|F\|_{L^\infty(\Omega;w)} \|u\|_{QH_0^1(\Omega;w)} \|v\|_{QH_0^1(\Omega;w)}. \end{aligned}$$

Combining these estimates we find,

$$|\mathscr{L}(u,v)| \leq \left(1 + \sqrt{N}\|\mathbf{H}\|_{L^\infty(\Omega)} + \|F\|_{L^\infty(\Omega)}\right) \|u\|_{QH_0^1(\Omega;w)} \|v\|_{QH_0^1(\Omega;w)}.$$

Setting $C = 1 + \sqrt{N}\|\mathbf{H}\|_{L^\infty(\Omega)} + \|F\|_{L^\infty(\Omega)} > 0$ gives

$$|\mathscr{L}(u,v)| \leq C\|u\|_{QH_0^1(\Omega;w)} \|v\|_{QH_0^1(\Omega;w)}.$$

□

The bilinear form (4.4) fails to satisfy the coercivity property in the Lax-Milgram Theorem. However, it is almost coercive meaning $\mathscr{L}$ is only a compact operator away from having an associated coercive bilinear form. It is worth noting that if $\mathbf{H} = \mathbf{0}$ and $F = 0$ almost everywhere in Ω then $\mathscr{L}$ satisfies the coercivity criterion by Lemma 5.2.

Lemma 4.6. *There are constants $c_1, c_2 > 0$ such that*

$$\mathscr{L}(u,u) \geq c_1\|u\|^2_{QH_0^1(\Omega;w)} - c_2\|u\|^2_{L^2(\Omega;w)}$$

for all $u \in QH_0^1(\Omega;w)$.

Proof. Fix $u \in QH_0^1(\Omega;w)$. Then,

$$\begin{aligned} \mathscr{L}(u,u) &\geq \int_\Omega |\sqrt{Q}\nabla u|^2\, dx - \left|\int_\Omega u\mathbf{H}\mathbf{T}u\, w dx\right| - \left|\int_\Omega uFu\, w dx\right| \\ &\geq \int_\Omega |\sqrt{Q}\nabla u|^2\, dx - \int_\Omega |u||\mathbf{H}||\mathbf{T}u|\, w dx - \int_\Omega |Fu^2|\, w dx \\ &\geq \frac{1}{(\tilde{c}+1)^2}\|u\|^2_{QH_0^1(\Omega)} - I - II \end{aligned}$$

by Lemma 5.2. We estimate terms I and II separately beginning with I. By Lemma 4.1, Theorem 2.6, and Lemma 2.4 we have that for any $\varepsilon > 0$,

$$\begin{aligned}
I &\leq \sqrt{N}\int_{\Omega} |u\sqrt{w}||\mathbf{H}||\sqrt{Q}\nabla u|\, dx \\
&\leq \sqrt{N}\|\mathbf{H}\|_{L^{\infty}(\Omega;w)}\|u\|_{L^2(\Omega;w)}\|\sqrt{Q}\nabla u\|_{L^2(\Omega)} \\
&\leq \sqrt{N}\|\mathbf{H}\|_{L^{\infty}(\Omega;w)}\,\varepsilon\|u\|^2_{L^2(\Omega;w)} + \sqrt{N}\|\mathbf{H}\|_{L^{\infty}(\Omega;w)}\,\varepsilon^{-1}\|\sqrt{Q}\nabla u\|^2_{L^2(\Omega)} \\
&\leq \sqrt{N}\|\mathbf{H}\|_{L^{\infty}(\Omega;w)}\,\varepsilon\|u\|^2_{L^2(\Omega;w)} + \sqrt{N}\|\mathbf{H}\|_{L^{\infty}(\Omega;w)}\,\varepsilon^{-1}\|u\|^2_{QH^1_0(\Omega;w)} \\
&\leq \left(\frac{\|\mathbf{H}\|^2_{L^{\infty}(\Omega;w)}}{2} + \frac{N}{2}\right)\varepsilon\|u\|^2_{L^2(\Omega;w)} + \left(\frac{\|\mathbf{H}\|^2_{L^{\infty}(\Omega;w)}}{2} + \frac{N}{2}\right)\varepsilon^{-1}\|u\|^2_{QH^1_0(\Omega;w)}.
\end{aligned}$$

Choosing $\varepsilon = \left(\|\mathbf{H}\|^2_{L^{\infty}(\Omega;w)} + N\right)(\tilde{c}+1)^2 > 0$ then,

$$I \leq \frac{\left(\|\mathbf{H}\|^2_{L^{\infty}(\Omega;w)} + N\right)^2(\tilde{c}+1)^2}{2}\|u\|^2_{L^2(\Omega;w)} + \left(\frac{1}{2(\tilde{c}+1)^2}\right)\|u\|^2_{QH^1_0(\Omega;w)}.$$

Transitioning to II, Theorem 2.6 gives

$$\begin{aligned}
II &\leq \|F\|_{L^{\infty}(\Omega;w)}\|u^2 w\|_{L^1(\Omega)} \\
&= \|F\|_{L^{\infty}(\Omega;w)}\|u\|^2_{L^2(\Omega;w)}.
\end{aligned}$$

Combining these estimates we find,

$$\mathscr{L}(u,u) \geq \frac{1}{2(\tilde{c}+1)^2}\|u\|^2_{QH^1_0(\Omega;w)} - \left(\frac{\left(\|\mathbf{H}\|^2_{L^{\infty}(\Omega;w)} + N\right)^2(\tilde{c}+1)^2}{2} + \|F\|_{L^{\infty}(\Omega;w)}\right)\|u\|^2_{L^2(\Omega;w)}.$$

Setting $c_1 = \dfrac{1}{2(\tilde{c}+1)^2}$, $c_2 = \dfrac{\left(\|\mathbf{H}\|^2_{L^{\infty}(\Omega;w)} + N\right)^2(\tilde{c}+1)^2}{2} + \|F\|_{L^{\infty}(\Omega;w)}$ where $c_1, c_2 > 0$ gives

$$\mathscr{L}(u,u) \geq c_1\|u\|^2_{QH^1_0(\Omega;w)} - c_2\|u\|^2_{L^2(\Omega;w)}.$$

□

We take advantage of $\mathscr{L}$ satisfying almost coercivity by setting

$$\mathscr{B}(z,v) = \mathscr{L}(z,v) + c_2\int_{\Omega} zv\, w dx$$

for $z, v \in QH_0^1(\Omega; w)$. Since

$$\left| \int_\Omega zv \, w dx \right| \leq \|z\|_{QH_0^1(\Omega;w)} \|v\|_{QH_0^1(\Omega;w)},$$

$\mathscr{B}$ is bounded. More, by Lemma 4.6

$$\begin{aligned} \mathscr{B}(z,z) &= \mathscr{L}(z,z) + c_2 \|z\|^2_{L^2(\Omega;w)} \\ &\geq c_1 \|z\|^2_{QH_0^1(\Omega;w)} - c_2 \|z\|^2_{L^2(\Omega;w)} + c_2 \|z\|^2_{L^2(\Omega;w)} \\ &= c_1 \|z\|^2_{QH_0^1(\Omega;w)}. \end{aligned}$$

Hence, $\mathscr{B}$ is coercive and since the $QH_0^1(\Omega; w)$ norm is equivalent to the inner product (2.18), the Lax-Milgram Theorem gives for every $\mathscr{G} \in (QH_0^1(\Omega; w))^*$, there exists a unique z $\in QH_0^1(\Omega; w)$ so that

$$\mathscr{B}(z, v) = \mathscr{G}(v)$$

holds for each $v \in Lip_0(\Omega)$. This defines a mapping

$$S : (QH_0^1(\Omega; w))^* \to QH_0^1(\Omega; w)$$

sending the bounded linear functional to the corresponding weak solution in $QH_0^1(\Omega; w)$. The mapping S is linear, continuous, bijective, and therefore invertible. That is, $S^{-1} : QH_0^1(\Omega; w) \to (QH_0^1(\Omega; w))^*$ is also linear, continuous, and bijective. Define the mapping $I_{L^2} : L^2(\Omega; w) \to (L^2(\Omega; w))^* \subset (QH_0^1(\Omega; w))^*$ by setting, for $h \in L^2(\Omega; w)$,

$$I_{L^2}(h)(s) = \int_\Omega hs \, w dx$$

for every $s \in L^2(\Omega; w)$. Note, I_{L^2} is continuous because it is linear and bounded. Since the projection mapping $\Pi : QH_0^1(\Omega; w) \to L^2(\Omega; w)$ is defined by

$$\Pi((u, \nabla u)) = u,$$

then for every $v \in QH_0^1(\Omega; w)$,

$$\int_\Omega uv \, w dx = I_{L^2} \Pi((u, \nabla u))(v).$$

As I_{L^2} is continuous and Π is compact, the composition $I_{L^2}\Pi$ is compact. This composition property enables the establishment of the following lemma.

Lemma 4.7. *Let $u \in QH_0^1(\Omega; w)$. Then, u is a solution of (4.4) if and only if u is a solution to the equation $(I - K)u = \alpha$ where I is the identity mapping, $K = c_2 S I_{L^2} \Pi$ is a compact operator, and $\alpha = S(\mathscr{F}(v)) \in QH_0^1(\Omega; w)$.*

Proof. Suppose $u \in QH_0^1(\Omega; w)$ is a solution of (4.4). Then, u is a solution of

$$\mathscr{B}(u, v) = \mathscr{L}(u, v) + c_2 \int_\Omega uv\, w dx = \int_\Omega fv\, w dx + c_2 \int_\Omega uv\, w dx = \mathscr{G}(v). \quad (4.6)$$

Notice, (4.6) is equivalent to

$$\mathscr{B}(u, v) - c_2 I_{L^2} \Pi((u, \nabla u))(v) = \int_\Omega fv\, w dx = \mathscr{F}(v). \quad (4.7)$$

Applying the mapping $S : (QH_0^1(\Omega; w))^* \to QH_0^1(\Omega; w)$ to (4.7) gives

$$S(\mathscr{B}(u, v)) - c_2 S(I_{L^2} \Pi((u, \nabla u))(v)) = S(\mathscr{F}(v)) \quad (4.8)$$

where $S(\mathscr{B}(u, v)) = u$ since u is a solution of (4.6) and $S(\mathscr{F}(v)) = \alpha$ for some $\alpha \in QH_0^1(\Omega; w)$. Since S is continuous and $I_{L^2} \Pi$ is compact, we can define the compact operator $K : QH_0^1(\Omega; w) \to QH_0^1(\Omega; w)$ by setting $K = c_2 S I_{L^2} \Pi$. That is,

$$Ku = c_2 S(I_{L^2} \Pi((u, \nabla u))(v))$$

and so, u is a solution to the equation $(I - K)u = \alpha$. If u is a solution of $(I - K)u = \alpha$, the reverse argument shows u is a solution of (4.4).

□

By the Fredholm Alternative, either the homogeneous equation

$$(I - K)u = 0$$

has a nontrivial solution $u \in QH_0^1(\Omega; w)$ or for each $\alpha \in QH_0^1(\Omega; w)$, the equation

$$(I - K)u = \alpha$$

has a uniquely determined solution $u \in QH_0^1(\Omega; w)$. In our context, the kernel of the operator $I - K$ is trivial and so, the former is impossible. To prove this, it suffices to show the only weak solution to the homogeneous Dirichlet problem is $(u, \nabla u) = (0, \mathbf{0})$. This argument, however, requires the choice of a test function $v \in QH_0^1(\Omega; w)$ to construct a contradiction. Therefore, we demonstrate that a weak solution of (4.4) not only holds for all $v \in Lip_0(\Omega)$, but also holds for all $v \in QH_0^1(\Omega; w)$ by the following density argument.

Lemma 4.8. *Suppose that the Sobolev inequality* (2.19) *holds. If* $u \in QH_0^1(\Omega; w)$ *is a weak solution of* (4.4)*, then*

$$\mathscr{L}(u, v) = \mathscr{F}(v)$$

holds for all $v \in QH_0^1(\Omega; w)$.

Proof. Let $(u, \nabla u)$ and $(h, \nabla h) \in QH_0^1(\Omega; w)$ where the former is a weak solution of (4.4). Then, there is a representative sequence $\{f_n\} \in Lip_0(\Omega)$ such that

$$\|f_n - h\|_{L^2(\Omega;w)} \to 0 \text{ as } n \to \infty$$

and

$$\|\sqrt{Q}\nabla f_n - \sqrt{Q}\nabla h\|_{L^2(\Omega)} \to 0 \text{ as } n \to \infty.$$

Fix $n \in \mathbb{N}$, then

$$|\mathscr{L}(u, h) - \mathscr{L}(u, f_n)| \leq I + II + III,$$

where

$$\begin{aligned}
I &= \int_\Omega |\sqrt{Q}\nabla u(\sqrt{Q}\nabla h - \sqrt{Q}\nabla f_n)|\, dx \leq \|\sqrt{Q}\nabla u\|_{L^2(\Omega)}\|\sqrt{Q}\nabla h - \sqrt{Q}\nabla f_n\|_{L^2(\Omega)};\\
II &= \int_\Omega |(h - f_n)\mathbf{HT}u|\, w dx \leq \sqrt{N}\|\mathbf{H}\|_{L^\infty(\Omega;w)}\|\sqrt{Q}\nabla u\|_{L^2(\Omega)}\|h - f_n\|_{L^2(\Omega;w)};\\
III &= \int_\Omega |(h - f_n)Fu|\, w dx \leq \|F\|_{L^\infty(\Omega;w)}\|u\|_{L^2(\Omega;w)}\|h - f_n\|_{L^2(\Omega;w)}.
\end{aligned}$$

More,

$$\begin{aligned}
|\mathscr{F}(h) - \mathscr{F}(f_n)| &= \left|\int_\Omega f(h - f_n)\, w dx\right|\\
&\leq \|f\|_{L^2(\Omega)}\|h - f_n\|_{L^2(\Omega;w)}.
\end{aligned}$$

Noting $\mathscr{L}(u, f_n) = \mathscr{F}(f_n)$ for each $n \in \mathbb{N}$, sending $n \to \infty$ gives

$$\mathscr{L}(u, h) = \lim_{n\to\infty} \mathscr{F}(f_n) = \mathscr{F}(h).$$

□

Lemma 4.9. *If $u \in QH_0^1(\Omega; w)$ is a weak solution of the Dirichlet problem*

$$\begin{cases} -\dfrac{1}{w} Div\,(Q\nabla u) + \mathbf{HT}u + Fu &= 0, \quad x \in \Omega; \\ u &= 0, \quad x \in \partial\Omega; \end{cases} \tag{4.9}$$

then $(u, \nabla u) = (0, \mathbf{0})$ in $QH_0^1(\Omega; w)$.

Proof. Let $(u, \nabla u) \in QH_0^1(\Omega; w)$ be a weak solution of (4.9) such that $(u, \nabla u) \neq (0, \mathbf{0})$. Then,

$$\mathscr{L}(u, v) = \int_\Omega \nabla u^T Q \nabla v \, dx + \int_\Omega v\mathbf{HT}u \, w dx + \int_\Omega vFu \, w dx = 0$$

holds for all $v \in QH_0^1(\Omega; w)$. Note, since $(u, \nabla u) \neq (0, \mathbf{0})$, Sobolev inequality (2.19) gives that $supp(|\sqrt{Q}\nabla u|)$ has positive measure. If $\mathbf{H} = \mathbf{0}$, $F = 0$, we let the test function $v = u$, then by Lemma 5.2,

$$0 = \mathscr{L}(u, u) = \int_\Omega |\sqrt{Q}\nabla u|^2 \, dx \geq \frac{1}{(\tilde{c}+1)^2} \|u\|^2_{QH_0^1(\Omega;w)}.$$

That is, $(u, \nabla u) = (0, \mathbf{0})$ by non-negativity of the norm. From this point onward, we will assume that

$$\|\mathbf{H}\|_{L^\infty(\Omega;w)} + \|F\|_{L^\infty(\Omega;w)} \neq 0.$$

Let the test function $v = u$. Since F is non-negative, then by Lemma 4.1 and Theorem 2.6,

$$\begin{aligned} \int_\Omega |\sqrt{Q}\nabla u|^2 \, dx &= -\int_\Omega u\mathbf{HT}u \, w dx - \int_\Omega Fu^2 \, w dx \\ &\leq \int_\Omega |u\sqrt{w}||\mathbf{H}||\mathbf{T}u| \, \sqrt{w} dx \\ &\leq \sqrt{N} \int_\Omega |u\sqrt{w}||\mathbf{H}||\sqrt{Q}\nabla u| \, dx \\ &\leq \sqrt{N}\|\mathbf{H}\|_{L^\infty(\Omega;w)} \|u\|_{L^2(\Omega;w)} \|\sqrt{Q}\nabla u\|_{L^2(\Omega).} \end{aligned}$$

Dividing by $\|\sqrt{Q}\nabla u\|_{L^2(\Omega)}$ and employing Sobolev inequality (2.19),

$$\begin{aligned} \left(\int_\Omega |\sqrt{Q}\nabla u|^2 \, dx \right)^{\frac{1}{2}} &\leq \sqrt{N}\|\mathbf{H}\|_{L^\infty(\Omega;w)} \|u\|_{L^2(\Omega;w)} \\ &\leq C\sqrt{N}\|\mathbf{H}\|_{L^\infty(\Omega;w)} \|\sqrt{Q}\nabla u\|_{L^2(\Omega)}. \end{aligned}$$

Dividing again by $\|\sqrt{Q}\nabla u\|_{L^2(\Omega)}$,

$$1 \leq C\sqrt{N}\|\mathbf{H}\|_{L^\infty(\Omega;w)}$$

giving a contradiction by (4.5). Therefore, $u = 0$ almost everywhere in Ω. Consequently, any weak solution of (4.9) is of the form $(u, \nabla u) = (0, \nabla u)$ for almost every $x \in \Omega$. Let $(0, \nabla u)$ be a weak solution of (4.9) then,

$$\mathscr{L}(u, v) = \int_\Omega \nabla u^T Q \nabla v \, dx = 0$$

holds for all $v \in QH_0^1(\Omega; w)$. Set $v = (0, \nabla u)$ then

$$\int_\Omega |\sqrt{Q}\nabla u|^2 \, dx = 0.$$

By a previous argument, this shows $\nabla u = \mathbf{0}$ almost everywhere in Ω. Therefore, $(u, \nabla u) = (0, \mathbf{0})$ is the only possible weak solution of (4.9).

□

Since the kernal of the operator $I - K$ is trivial, the Fredholm Alternative guarantees the existence of a unique solution $(u, \nabla u) \in QH_0^1(\Omega; w)$ satisfying $(I - K)u = \alpha$. By Lemma 4.7, we have proved the following theorem.

Theorem 4.10. *Let Q be a non-negative definite symmetric matrix-valued function in $\Omega \subset \mathbb{R}^n$ with the non-negative weight $w \in L^1_{loc}(\Omega)$ that satisfies (1.1). Suppose there exists a constant $C > 0$ such that*

$$\|\varphi\|_{L^2(\Omega;w)} \leq C\|\sqrt{Q}\nabla\varphi\|_{L^2(\Omega)}$$

for all $\varphi \in Lip_0(\Omega)$. Then, given $f \in L^2(\Omega)$, there is a unique weak solution $(u, \nabla u) \in QH_0^1(\Omega; w)$ of the Dirichlet problem

$$\begin{cases} -\dfrac{1}{w} Div(Q\nabla u) + \mathbf{HT}u + Fu &= f, \quad x \in \Omega; \\ u &= 0, \quad x \in \partial\Omega; \end{cases} \tag{4.10}$$

whenever $\mathbf{H}, F \in L^\infty(\Omega; w)$ and $\mathbf{T}$ is a vector of first order vector fields satisfying (4.2).

4.2.2 Exponentially Integrable Coefficients Case

Suppose Sobolev inequality (2.20) holds and let the coefficient functions $\mathbf{H} \in L^{\Psi_H}(\Omega; w)$ and $F \in L^{\Psi_F}(\Omega; w)$ where $\Psi_H(t) = e^{t^{\frac{2}{\gamma}}} - 1$ and $\Psi_F(t) = e^{t^{\frac{1}{\gamma}}} - 1$ for $0 < \gamma \leq \frac{q}{2}$. Notice, Lemma 4.7 was independent of the roughness of $\mathbf{H}$, F, and only dependent on the continuity of the mapping

$$S : (QH_0^1(\Omega; w))^* \to QH_0^1(\Omega; w)$$

and the compactness of the projection mapping

$$\Pi : QH_0^1(\Omega; w) \to QH_0^1(\Omega; w).$$

Therefore, we make the identical argument seen in the previous section to prove Theorem 4.16 after establishing the following estimate that is required in Lemma 4.12 and Lemma 4.13. In the development hereafter, constants K_1, K_2, K_3, and K_4 correspond to Lemma 4.11, Sobolev inequality (2.20), Theorem 5.1, and Lemma 2.22 respectively. Note, the properties of every Young Function used throughout this section is summarized in Table 1.

Lemma 4.11. *Let $u \in QH_0^1(\Omega; w)$. Then,*

$$\|u\mathbf{H}\|_{L^2(\Omega;w)} \leq C\|u\|_{L^\Phi(\Omega;w)}\|\mathbf{H}\|_{L^{\Psi_H}(\Omega;w)}$$

where $\Phi(t) = t^2 log(e - 1 + t)^\gamma$ for $0 < \gamma \leq \frac{q}{2}$ and $C > 1$.

Proof. Let $E(t) = t\log(e - 1 + t)^\gamma$ then $\bar{E}(t) = e^{t^{\frac{1}{\gamma}}} - 1$ and set $A(t) = t^2\log(e - 1 + t^2)^\gamma$. Since $E(t) = A(t^{\frac{1}{2}})$ and $\bar{E}(t) = \Psi_H(t^{\frac{1}{2}})$, Proposition 2.19 demonstrates

$$\begin{aligned}\|u^2\|_{L^E(\Omega;w)} &= \|u\|^2_{L^A(\Omega;w)};\\ \|\mathbf{H}^2\|_{L^{\bar{E}}(\Omega;w)} &= \|\mathbf{H}\|^2_{L^{\Psi_H}(\Omega;w)}.\end{aligned}$$

where $\mathbf{H}^2$ denotes the dot product of $\mathbf{H}$ with itself. Notice, $A(t) \leq t^2\log((e - 1 + t)^2)^\gamma = 2^\gamma t^2\log(e - 1 + t)^\gamma = 2^\gamma\Phi(t)$. Therefore, by Theorem 2.18 and Proposition 2.23

$$\begin{aligned}\|u\mathbf{H}\|^2_{L^2(\Omega;w)} &= \int_\Omega |u|^2|\mathbf{H}|^2\, w dx\\ &\leq 2\|u^2\|_{L^E(\Omega;w)}\|\mathbf{H}^2\|_{L^{\bar{E}}(\Omega;w)}\\ &\leq 2^{\gamma+1}\|u\|^2_{L^\Phi(\Omega;w)}\|\mathbf{H}\|^2_{L^{\Psi_H}(\Omega;w)}.\end{aligned}$$

Taking the square root of both sides and setting $C = 2^{\frac{\gamma+1}{2}} > 1$ gives

$$\|u\mathbf{H}\|_{L^2(\Omega;w)} \leq C\|u\|_{L^\Phi(\Omega;w)}\|\mathbf{H}\|_{L^{\Psi_H}(\Omega;w)}.$$

□

Lemma 4.12. *There exists a constant $C > 0$ such that*

$$|\mathscr{L}(u,v)| \leq C\|u\|_{QH_0^1(\Omega;w)}\|v\|_{QH_0^1(\Omega;w)}$$

for all $u, v \in QH_0^1(\Omega; w)$.

Proof. Let $u, v \in QH_0^1(\Omega; w)$. Then, by Theorem 2.5

$$\begin{aligned}
|\mathscr{L}(u,v)| &\leq \int_\Omega |\sqrt{Q}\nabla u\sqrt{Q}\nabla v|\,dx + \int_\Omega |v\mathbf{H}\mathbf{T}u|\,wdx + \int_\Omega |vFu|\,wdx \\
&\leq \|\sqrt{Q}\nabla u\|_{L^2(\Omega)}\|\sqrt{Q}\nabla v\|_{L^2(\Omega)} + \int_\Omega |v||\mathbf{H}||\mathbf{T}u|\,wdx + \int_\Omega |v||F||u|\,wdx \\
&\leq \|u\|_{QH_0^1(\Omega;w)}\|v\|_{QH_0^1(\Omega;w)} + I + II.
\end{aligned}$$

We estimate terms I and II separately beginning with the former. By Lemma 4.1, Theorem 2.5, Lemma 4.11, Proposition 2.23, Lemma 7.4, and Sobolev inequality (2.20), we have that

$$\begin{aligned}
I &= \int_\Omega |v|\mathbf{H}||\mathbf{T}u|\,wdx \\
&\leq \sqrt{N}\int_\Omega |v\sqrt{w}\mathbf{H}||\sqrt{Q}\nabla u|\,dx \\
&\leq \sqrt{N}\|v\mathbf{H}\|_{L^2(\Omega;w)}\|\sqrt{Q}\nabla u\|_{L^2(\Omega)} \\
&\leq \sqrt{N}K_1\|\mathbf{H}\|_{L^{\Psi_{\mathrm{H}}}(\Omega;w)}\|v\|_{L^{\Phi}(\Omega;w)}\|u\|_{QH_0^1(\Omega;w)} \\
&\leq \sqrt{N}K_1 2^q\|\mathbf{H}\|_{L^{\Psi_{\mathrm{H}}}(\Omega;w)}\|v\|_{L^{\Psi}(\Omega;w)}\|u\|_{QH_0^1(\Omega;w)} \\
&\leq \sqrt{N}K_1K_2 2^q\|\mathbf{H}\|_{L^{\Psi_{\mathrm{H}}}(\Omega;w)}\|u\|_{QH_0^1(\Omega;w)}\|v\|_{QH_0^1(\Omega;w)}.
\end{aligned}$$

Before estimating II, notice for $c = \dfrac{2^q}{\log(e-1)^q}$ that

$$\Psi^{-1}(t)\Psi_F^{-1}(t) \leq \bar{\Psi}^{-1}(ct)$$

by Lemma 7.5 for all $t \geq 0$. Therefore, Theorem 5.1 with $\lambda_1 = \lambda_2 = 1$ gives

$$\|vF\|_{L^{\bar{\Psi}}(\Omega;w)} \leq K_3\|F\|_{L^{\Psi_F}(\Omega;w)}\|v\|_{L^{\Psi}(\Omega;w)}.$$

Transitioning to II, Theorem 2.18 and Sobolev inequality (2.20) give

$$\begin{aligned} II &= \int_\Omega |v||F||u|\, w dx \\ &\leq 2\|u\|_{L^\Psi(\Omega;w)}\|vF\|_{L^{\bar\Psi}(\Omega;w)} \\ &\leq 2K_3\|F\|_{L^{\Psi_F}(\Omega;w)}\|u\|_{L^\Psi(\Omega;w)}\|v\|_{L^\Psi(\Omega;w)} \\ &\leq 2K_2^2K_3\|F\|_{L^{\Psi_F}(\Omega;w)}\|u\|_{QH_0^1(\Omega;w)}\|v\|_{QH_0^1(\Omega;w)}. \end{aligned}$$

Choosing

$$C = 1 + \sqrt{N}K_1K_2 2^q\|\mathbf{H}\|_{L^{\Psi_{\mathrm{H}}}(\Omega;w)} + 2K_2^2K_3\|F\|_{L^{\Psi_F}(\Omega;w)} > 0$$

then

$$|\mathscr{L}(u,v)| \leq C\|u\|_{QH_0^1(\Omega;w)}\|v\|_{QH_0^1(\Omega;w)}.$$

□

Lemma 4.13. *There are constants $c_1, c_2 > 0$ such that*

$$\mathscr{L}(u,u) \geq c_1\|u\|^2_{QH_0^1(\Omega;w)} - c_2\|u\|^2_{L^2(\Omega;w)}$$

for all $u \in QH_0^1(\Omega;w)$.

Proof. Fix $u \in QH_0^1(\Omega;w)$. By the reverse triangle inequality,

$$\begin{aligned} \mathscr{L}(u,u) &\geq \int_\Omega |\sqrt{Q}\nabla u|^2\, dx - \Big|\int_\Omega u\mathbf{HT}u\, w dx\Big| - \Big|\int_\Omega uFu\, w dx\Big| \\ &\geq \int_\Omega |\sqrt{Q}\nabla u|^2\, dx - \int_\Omega |u||\mathbf{H}||\mathbf{T}u|\, w dx - \int_\Omega |Fu^2|\, w dx \\ &\geq \frac{1}{(\tilde c+1)^2}\|u\|^2_{QH_0^1(\Omega;w)} - I - II \end{aligned}$$

by Lemma 5.2. We estimate terms I and II separately beginning with I. Let $E(t) = t\log(e-1+t)^\gamma$ and $B(t) = t\log(e-1+t)^q$. Since $0 < \gamma \leq \frac{q}{2}$, we have that

$$t^{\frac{1}{2}}(B^{-1}(t))^{\frac{1}{2}} \leq E^{-1}(ct)$$

for all $t \geq 1$ when $c = 1$; see Lemma 7.6 and definition 7.3. Therefore, by Lemma 2.22

$$\begin{aligned}\|u^2\|^{\frac{1}{2}}_{L^E(\Omega;w)} &\leq \left(K_4\varepsilon_1^{-1}\|u^2\|_{L^1(\Omega;w)} + K_4\varepsilon_1\|u^2\|_{L^B(\Omega;w)}\right)^{\frac{1}{2}} \\ &\leq \left(K_4\varepsilon_1^{-1}\|u^2\|^{\frac{1}{2}}_{L^1(\Omega;w)} + K_4\varepsilon_1\|u^2\|^{\frac{1}{2}}_{L^B(\Omega;w)}\right).\end{aligned}$$

Set $C(t) = t^2\log(e-1+t^2)^q$ then $B(t) = C(t^{\frac{1}{2}})$ and Proposition 2.19 gives

$$\begin{aligned}\|u^2\|^{\frac{1}{2}}_{L^1(\Omega;w)} &= \|u\|_{L^2(\Omega;w)}; \\ \|u^2\|^{\frac{1}{2}}_{L^B(\Omega;w)} &= \|u\|_{L^C(\Omega;w)}.\end{aligned}$$

Notice, $C(t) \leq t^2\log((e-1+t)^2)^q = 2^q\Psi(t)$ and so, by Proposition 2.23

$$\|u\|_{L^C(\Omega;w)} \leq 2^q\|u\|_{L^\Psi(\Omega;w)} = K_5\|u\|_{L^\Psi(\Omega;w)}.$$

That is,

$$\|u^2\|^{\frac{1}{2}}_{L^E(\Omega;w)} \leq K_4\varepsilon_1^{-1}\|u\|_{L^2(\Omega;w)} + K_4K_5\varepsilon_1\|u\|_{L^\Psi(\Omega;w)}.$$

By Lemma 4.1, Theorem 2.5, Theorem 2.18, Proposition 2.19, Corollary 2.4, and Sobolev inequality (2.20), we have that

$$\begin{aligned}I &= \int_\Omega |u||\mathbf{H}||\mathbf{T}u|\,wdx \\ &\leq \sqrt{N}\int_\Omega |u\sqrt{w}\mathbf{H}||\sqrt{Q}\nabla u|\,dx \\ &\leq \sqrt{N}\|u\mathbf{H}\|_{L^2(\Omega;w)}\|\sqrt{Q}\nabla u\|_{L^2(\Omega)} \\ &= \sqrt{N}\left(\int_\Omega |u|^2|\mathbf{H}|^2\,wdx\right)^{\frac{1}{2}}\|\sqrt{Q}\nabla u\|_{L^2(\Omega)} \\ &\leq \sqrt{2N}\|\mathbf{H}^2\|^{\frac{1}{2}}_{L^{\bar{E}}(\Omega;w)}\|u^2\|^{\frac{1}{2}}_{L^E(\Omega;w)}\|\sqrt{Q}\nabla u\|_{L^2(\Omega)} \\ &= \sqrt{2N}\|\mathbf{H}\|_{L^{\Psi_{\mathrm{H}}}(\Omega;w)}\|u^2\|^{\frac{1}{2}}_{L^E(\Omega;w)}\|\sqrt{Q}\nabla u\|_{L^2(\Omega)} \\ &\leq \sqrt{2N}\|\mathbf{H}\|_{L^{\Psi_{\mathrm{H}}}(\Omega;w)}\left(K_4\varepsilon_1^{-1}\|u\|_{L^2(\Omega;w)} + K_4K_5\varepsilon_1\|u\|_{L^\Psi(\Omega;w)}\right)\|\sqrt{Q}\nabla u\|_{L^2(\Omega)} \\ &\leq \sqrt{2N}\|\mathbf{H}\|_{L^{\Psi_{\mathrm{H}}}(\Omega;w)}K_4\varepsilon_1^{-1}\left(\varepsilon_2\|\sqrt{Q}\nabla u\|^2_{L^2(\Omega)} + \varepsilon_2^{-1}\|u\|^2_{L^2(\Omega;w)}\right) \\ &+ \sqrt{2N}\|\mathbf{H}\|_{L^{\Psi_{\mathrm{H}}}(\Omega;w)}K_2K_4K_5\varepsilon_1\|u\|^2_{QH^1_0(\Omega;w)} \\ &\leq \frac{K_4}{2}\left(2N + \|\mathbf{H}\|^2_{L^{\Psi_{\mathrm{H}}}}\right)\left(\varepsilon_1^{-1}\varepsilon_2 + K_2K_5\varepsilon_1\right)\|u\|^2_{QH^1_0(\Omega;w)} + \frac{K_4}{2}\left(N + \|\mathbf{H}\|^2_{L^{\Psi_{\mathrm{H}}}}\right)\varepsilon_2^{-1}\|u\|^2_{L^2(\Omega;w)},\end{aligned}$$

where $\varepsilon_1, \varepsilon_2 > 0$ and will be chosen momentarily. Transitioning to II and employing Theorem 2.18, Lemma 2.22, Proposition 2.19, Proposition 2.23, Corollary 2.4, and Sobolev inequality (2.20) gives

$$\begin{aligned}
II &= \int_\Omega |Fu^2|\, w dx \\
&\leq 2\|F\|_{L^{\Psi_F}(\Omega;w)} \|u^2\|_{L^E(\Omega;w)} \\
&\leq 2\|F\|_{L^{\Psi_F}(\Omega;w)} \left(K_4 \varepsilon_3^{-1} \|u\|^2_{L^2(\Omega;w)} + K_4 K_5 \varepsilon_3 \|u\|^2_{L^\Psi(\Omega;w)} \right) \\
&\leq \left(\|F\|^2_{L^{\Psi_F}(\Omega;w)} + K_4^2 \right) \varepsilon_3^{-1} \|u\|^2_{L^2(\Omega;w)} + \left(\|F\|^2_{L^{\Psi_F}(\Omega;w)} + K_4^2 \right) K_2^2 K_5 \varepsilon_3 \|u\|^2_{QH_0^1(\Omega;w)}.
\end{aligned}$$

Choose $\varepsilon_1, \varepsilon_2$, and $\varepsilon_3 > 0$ such that

$$\begin{aligned}
\varepsilon_1 &= \frac{2}{K_2 K_4 K_5 \left(2N + \|\mathbf{H}\|^2_{L^{\Psi_\mathrm{H}}} \right) \mu_1 (\tilde{c}+1)^2}; \\
\varepsilon_2 &= \frac{2\varepsilon_1}{K_4 \left(2N + \|\mathbf{H}\|^2_{L^{\Psi_\mathrm{H}}} \right) \mu_2 (\tilde{c}+1)^2}; \\
\varepsilon_3 &= \frac{1}{K_2^2 K_5 \left(\|F\|^2_{L^{\Psi_F}(\Omega;w)} + K_4^2 \right) \mu_3 (\tilde{c}+1)^2}
\end{aligned}$$

where $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ satisfy

$$1 - \frac{1}{\mu_1} - \frac{1}{\mu_2} - \frac{1}{\mu_3} = \mu > 0.$$

Setting $c_1, c_2 > 0$ as follows completes the proof

$$\begin{aligned}
c_1 &= \frac{1}{\mu(\tilde{c}+1)^2}; \\
c_2 &= K_4 \left(\frac{2N + \|\mathbf{H}\|^2_{L^{\Psi_\mathrm{H}}}}{2} \right) \varepsilon_1^{-1} + \left(\|F\|^2_{L^{\Psi_F}(\Omega;w)} + K_4^2 \right) \varepsilon_2^{-1}.
\end{aligned}$$

□

Lemma 4.14. *Suppose that the Sobolev inequality* (2.20) *holds. If* $u \in QH_0^1(\Omega; w)$ *is a weak solution of* (4.4)*, then*

$$\mathscr{L}(u, v) = \mathscr{F}(v)$$

holds for all $v \in QH_0^1(\Omega; w)$.

Proof. Let $(u, \nabla u) \in QH_0^1(\Omega; w)$ be a weak solution of (4.4) and let $(h, \nabla h) \in QH_0^1(\Omega; w)$. Then, there is a representative sequence $\{f_n\} \in Lip_0(\Omega; w)$ such that

$$\|f_n - h\|_{L^2(\Omega;w)} \to 0 \text{ as } n \to \infty$$

and

$$\|\sqrt{Q}\nabla f_n - \sqrt{Q}\nabla h\|_{L^2(\Omega)} \to 0 \text{ as } n \to \infty.$$

Fix $n \in \mathbb{N}$, then

$$|\mathscr{L}(u, h) - \mathscr{L}(u, f_n)| \leq I + II + III$$

where

$$\begin{aligned} I &= \int_\Omega |\sqrt{Q}\nabla u(\sqrt{Q}\nabla h - \sqrt{Q}\nabla f_n)|\, dx \\ &\leq \|\sqrt{Q}\nabla u\|_{L^2(\Omega)} \|\sqrt{Q}\nabla h - \sqrt{Q}\nabla f_n\|_{L^2(\Omega)}; \end{aligned}$$

$$\begin{aligned} II &= \int_\Omega |(h - f_n)\mathbf{H}\mathbf{T}u|\, w dx \\ &\leq \sqrt{N} \int_\Omega |(h - f_n)\mathbf{H}||\sqrt{Q}\nabla u|\, w dx \\ &\leq \sqrt{N} \|(h - f_n)\mathbf{H}\|_{L^2(\Omega;w)} \|\sqrt{Q}\nabla u\|_{L^2(\Omega)} \\ &\leq \sqrt{N} C \|\mathbf{H}\|_{L^{\Psi_{\mathrm{H}}}(\Omega;w)} \|(h - f_n)\|_{L^{\Phi}(\Omega;w)} \|\sqrt{Q}\nabla u\|_{L^2(\Omega)} \\ &\leq \sqrt{N} C \|\mathbf{H}\|_{L^{\Psi_{\mathrm{H}}}(\Omega;w)} \|(h - f_n)\|_{L^{\Psi}(\Omega;w)} \|\sqrt{Q}\nabla u\|_{L^2(\Omega)}; \\ &\leq \sqrt{N} C K_2 \|\mathbf{H}\|_{L^{\Psi_{\mathrm{H}}}(\Omega;w)} \|\sqrt{Q}\nabla h - \sqrt{Q}\nabla f_n\|_{L^2(\Omega)} \|\sqrt{Q}\nabla u\|_{L^2(\Omega)}; \end{aligned}$$

$$\begin{aligned} III &= \int_\Omega |(h - f_n)Fu|\, w dx \\ &\leq 2\|(h - f_n)F\|_{L^{\bar{\Psi}(\Omega;w)}} \|u\|_{L^{\Psi}(\Omega;w)} \\ &\leq 2K_3 \|F\|_{L^{\Psi_F}(\Omega;w)} \|h - f_n\|_{L^{\Psi}(\Omega;w)} \|u\|_{L^{\Psi}(\Omega;w)} \\ &\leq 2K_2K_3 \|F\|_{L^{\Psi_F}(\Omega;w)} \|\sqrt{Q}\nabla h - \sqrt{Q}\nabla f_n\|_{L^2(\Omega)} \|u\|_{L^{\Psi}(\Omega;w)}. \end{aligned}$$

Additionally,

$$|\mathscr{F}(h) - \mathscr{F}(f_n)| = \Big| \int_\Omega f(h - f_n)\, w dx \Big| \\ \leq \|f\|_{L^2(\Omega;w)} \|h - f_n\|_{L^2(\Omega;w)}.$$

Since $\mathscr{L}(u, f_n) = \mathscr{F}(f_n)$ for each $n \in \mathbb{N}$, sending $n \to \infty$ gives

$$\mathscr{L}(u, h) = \lim_{n\to\infty} \mathscr{F}(f_n) = \mathscr{F}(h).$$

□

Lemma 4.15. *If $u \in QH_0^1(\Omega; w)$ is a weak solution of the Dirichlet problem*

$$\begin{cases} -\dfrac{1}{w} Div(Q\nabla u) + \mathbf{HT}u + Fu &= 0, \quad x \in \Omega; \\ u &= 0, \quad x \in \partial\Omega; \end{cases} \tag{4.11}$$

then $(u, \nabla u) = (0, \mathbf{0})$ in $QH_0^1(\Omega; w)$.

Proof. Let $(u, \nabla u) \in QH_0^1(\Omega; w)$ be a weak solution of (4.11) such that $(u, \nabla u) \neq (0, \mathbf{0})$. Then,

$$\mathscr{L}(u, v) = \int_\Omega \nabla u^T Q \nabla v \, dx + \int_\Omega v \mathbf{HT} u \, w dx + \int_\Omega v F u \, w dx = 0$$

holds for all $v \in QH_0^1(\Omega; w)$. Note, since $(u, \nabla u) \neq (0, \mathbf{0})$ Sobolev inequality (2.20) gives that $supp(|\sqrt{Q}\nabla u|)$ has positive measure. If $\mathbf{H} = \mathbf{0}$, F = 0, we let $u = v$, then by Lemma 5.2,

$$0 = \mathscr{L}(u, u) = \int_\Omega |\sqrt{Q}\nabla u|^2 \, dx \geq \frac{1}{(\tilde{c} + 1)^2} \|u\|^2_{QH_0^1(\Omega;w)}.$$

That is, $(u, \nabla u) = (0, \mathbf{0})$ by non-negativity of the norm. From this point onward, we will assume that

$$\|\mathbf{H}\|_{L^{\Psi_{\mathrm{H}}}(\Omega;w)} + \|F\|_{L^{\Psi_F}(\Omega;w)} \neq 0.$$

For each $k > 0$, $(v, \nabla v) = ((u - k)_+, \mathbb{1}_{\{u>k\}} \nabla u)$ is a valid test function for u; see [1]. Note, $\nabla v = \nabla u$ on the support of ∇v and since F is non-negative, then by Lemma 4.1 and Theorem 2.5,

$$\begin{aligned}\int_\Omega |\sqrt{Q}\nabla v|^2\,dx &= -\int_\Omega v\mathbf{HT}u\,w dx - \int_\Omega vFu\,w dx \\ &= \int_\Omega v\mathbf{HT}v\,w dx - \int_\Omega Fv^2\,w dx - k\int_\Omega vF\,w dx \\ &\le \int_\Omega |v\sqrt{w}\mathbf{H}||\mathbf{T}v|\sqrt{w}\,dx \\ &\le \sqrt{N}\|v\mathbf{H}\|_{L^2(\Omega;w)}\|\sqrt{Q}\nabla v\|_{L^2(\Omega)}.\end{aligned}$$

Dividing by $\|\sqrt{Q}\nabla v\|_{L^2(\Omega)}$,

$$\left(\int_\Omega |\sqrt{Q}\nabla v|^2\,dx\right)^{\frac{1}{2}} \le \sqrt{N}\|v\mathbf{H}\|_{L^2(\Omega;w)}. \tag{4.12}$$

By Lemma 4.11

$$\|v\mathbf{H}\|^2_{L^2(\Omega;w)} \le 2^{\gamma+1}\|\mathbf{H}\|^2_{L^{\Psi_{\mathrm{H}}}(\Omega;w)}\|v\cdot 1\|^2_{L^\Phi(\Omega;w)}$$

and with $\varphi^{-1}(t) = \log(e-1+t)^{\frac{q-\gamma}{2}}$, $\varphi(t) = e^{t^{\frac{2}{q-\gamma}}} - 1$ we have that

$$\Psi^{-1}(t)\varphi^{-1}(t) \le \Phi^{-1}(ct)$$

for all $t \ge 0$ when $c = 1$. Theorem 5.1 with $\lambda_1 = \lambda_2 = 1$ and Proposition 2.24 then show

$$\|v\cdot 1\|_{L^\Phi(\Omega;w)} \le K_6\|v\|_{L^\Psi(\Omega;w)}\|\mathbb{1}_s\|_{L^\varphi(\Omega;w)},$$

where $\|\mathbb{1}_s\|_{L^\varphi(\Omega;w)} = \dfrac{1}{\log\left(e-1+\dfrac{1}{w(S)}\right)^{\frac{q-\gamma}{2}}}$ and $S = supp(|\sqrt{Q}\nabla v|)$. Thus,

$$\begin{aligned}\|v\mathbf{H}\|^2_{L^2(\Omega;w)} &\le 2^{\gamma+1}\|\mathbf{H}\|^2_{L^{\Psi_{\mathrm{H}}}(\Omega;w)}\|v\|^2_{L^\Psi(\Omega;w)}\|\mathbb{1}_s\|^2_{L^\varphi(\Omega;w)} \\ &\le 2^{\gamma+1}K_2^2K_6\|\mathbf{H}\|^2_{L^{\Psi_{\mathrm{H}}}(\Omega;w)}\|\sqrt{Q}\nabla v\|^2_{L^2(\Omega;w)}\|\mathbb{1}_s\|^2_{L^\varphi(\Omega;w)}.\end{aligned} \tag{4.13}$$

Furthering estimate (4.12) with (4.13) and Proposition 2.3

$$\left(\int_\Omega |\sqrt{Q}\nabla v|^2\,dx\right)^{\frac{1}{2}} \leq 2^{\frac{\gamma+1}{2}}\sqrt{N}K_2\sqrt{K_6}\|\mathbf{H}\|_{L^{\Psi_{\mathrm{H}}}(\Omega;w)}\|\sqrt{Q}\nabla v\|_{L^2(\Omega;w)}\|\mathbb{1}_s\|_{L^\varphi(\Omega;w)}$$
$$\leq \left(\frac{2^{\gamma+1}+N}{2}\right)K_2\sqrt{K_6}\|\mathbf{H}\|_{L^{\Psi_{\mathrm{H}}}(\Omega;w)}\|\sqrt{Q}\nabla v\|_{L^2(\Omega;w)}\|\mathbb{1}_s\|_{L^\varphi(\Omega;w)}.$$

Dividing by $\|\sqrt{Q}\nabla v\|_{L^2(\Omega;w)}$ and setting $\beta = \left(\frac{2^{\gamma+1}+N}{2}\right)K_2\sqrt{K_6}\|\mathbf{H}\|_{L^{\Psi_{\mathrm{H}}}(\Omega;w)}$, we find

$$1 \leq \frac{\beta}{\log\left(e-1+\frac{1}{w(S)}\right)^{\frac{q-\gamma}{2}}}.$$

Setting $\delta = \left(\frac{1}{\beta}\right)^{\frac{2}{q-\gamma}} > 0$ and isolating for $w(S)$ we find

$$0 < \frac{1}{e^{\frac{1}{\delta}}} \leq w(S).$$

This estimate is independent of k and letting $k \to \sup_\Omega(u) > 0$ gives us that $v = 0$ almost everywhere while the $L^1(\Omega)$ function $|\sqrt{Q}\nabla v|^2$ has support with positive measure contradicting (4.12). Therefore,

$$\sup_\Omega(u) \leq 0.$$

Since $(v, \nabla v) = ((u+k)_-, \mathbb{1}_{\{u<-k\}}\nabla u)$ is another valid test function (see [1]), repeating the identical argument and letting $k \to \inf_\Omega(u) < 0$, we obtain the same contradiction and conclude

$$\inf_\Omega(u) \geq 0.$$

Therefore, $u = 0$ almost everywhere in Ω. Hence, any solution of (4.11) is of the form $(u, \nabla u) = (0, \nabla u) \in QH_0^1(\Omega)$. Let $(0, \nabla u)$ be a weak solution of (4.11) then,

$$\mathscr{L}(u, v) = \int_\Omega \nabla u^T Q \nabla v\,dx = 0$$

holds for all $v \in QH_0^1(\Omega; w)$. Set $v = (0, \nabla u)$ then

$$\int_\Omega |\sqrt{Q}\nabla u|^2 \, dx = 0.$$

By a previous argument, this shows $\nabla u = \mathbf{0}$ almost everywhere in Ω. Therefore, $(u, \nabla u) = (0, \mathbf{0})$ is the only possible weak solution of (4.11).

□

By the Fredholm Alternative and Lemma 4.7, we have proved the following theorem.

Theorem 4.16. *Let Q be a non-negative definite symmetric matrix-valued function in $\Omega \subset \mathbb{R}^n$ with the non-negative weight $w \in L^1_{loc}(\Omega)$ that satisfies* (1.1)*. Suppose there exists a constant $C > 0$ such that*

$$\|\varphi\|_{L^\Psi(\Omega;w)} \leq C\|\sqrt{Q}\nabla\varphi\|_{L^2(\Omega)}$$

for all $\varphi \in Lip_0(\Omega)$. Then, given $f \in L^2(\Omega)$, there is a unique weak solution $(u, \nabla u) \in QH_0^1(\Omega; w)$ of the Dirichlet problem

$$\begin{cases} -\dfrac{1}{w} Div\,(Q\nabla u) + \mathbf{HT}u + Fu &= f, \quad x \in \Omega; \\ u &= 0, \quad x \in \partial\Omega; \end{cases} \tag{4.14}$$

whenever $\mathbf{H} \in L^{\Psi_H}(\Omega; w), F \in L^{\Psi_F}(\Omega; w)$ where $\Psi_H(t) = e^{t^{\frac{2}{\gamma}}} - 1$, $\Psi_F(t) = e^{t^{\frac{1}{\gamma}}} - 1$ for $0 < \gamma \leq \frac{q}{2} \in \mathbb{R}^+$ and $\mathbf{T}$ is a vector of first order vector fields satisfying (4.2)*.*

5 Appendix A - Preliminary Estimates

The proof of the following theorem can be found in [11].

Theorem 5.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\Psi_1, \Psi_2, ..., \Psi_m$ and $\lambda_1, \lambda_2, ..., \lambda_m$ be Young functions and positive real numbers, respectively. Suppose that Φ is a Young function whose inverse satisfies*

$$\prod_{j=1}^{m} \Psi_j^{-1}(t)^{\lambda_j} \leq \Phi^{-1}(ct) \tag{5.1}$$

for some constant $c > 0$ and $t \geq \mu(X)^{-1}$. Then, for any collection of real-valued μ-measurable functions $f_1, f_2, ..., f_m$ on X, there exists a constant C that is independent of each f_j such that

$$\left\| \prod_{j=1}^{m} |f_j|^{\lambda_j} \right\|_{L^{\Phi}(X, \mathcal{A}, \mu)} \leq C \prod_{j=1}^{m} \|f_j\|^{\lambda_j}_{L^{\Psi_j}(X, \mathcal{A}, \mu)}. \tag{5.2}$$

Lemma 5.2. *Suppose Sobolev inequality* (2.19) *holds. Then, there are constants $c_1, c_2 > 0$ such that*

$$c_1 \|f\|_{QH_0^1(\Omega; w)} \leq \|f\|_{\mathcal{L}^2(Q;\Omega)} \leq c_2 \|f\|_{QH_0^1(\Omega; w)} \tag{5.3}$$

for all $f \in QH_0^1(\Omega; w)$ where $\|f\|_{\mathcal{L}^2(Q;\Omega)} = \left(\int_{\Omega} |\sqrt{Q} \nabla f|^2 \, dx \right)^{\frac{1}{2}}$.

Proof. Clearly $\|f\|_{\mathcal{L}^2(Q;\Omega)} \leq c_2 \|f\|_{QH_0^1(\Omega)}$ for $c_2 = 1$. By Sobolev inequality 2.19, we have that

$$\begin{aligned} \|f\|_{QH_0^1(\Omega; w)} &\leq \tilde{c} \left(\int_{\Omega} |\sqrt{Q} \nabla f|^2 \, dx \right)^{\frac{1}{2}} + \left(\int_{\Omega} |\sqrt{Q} \nabla f|^2 \, dx \right)^{\frac{1}{2}} \\ &= (\tilde{c} + 1) \, \|f\|_{\mathcal{L}^2(Q;\Omega)}. \end{aligned}$$

Setting $c_1 = \dfrac{1}{\tilde{c} + 1} > 0$ we find,

$$c_1 \|f\|_{QH_0^1(\Omega; w)} \leq \|f\|_{\mathcal{L}^2(Q;\Omega)} \leq c_2 \|f\|_{QH_0^1(\Omega; w)}.$$

□

6 Appendix B - Supporting Theorems

The proofs of these results can be found in [8] and [12].

6.1 Lax-Milgram Theorem Prerequisite Results

Theorem 6.1. *Let V be a normed vector space. A linear functional $\mathscr{F} : V \to \mathbb{R}$ is a continuous if and only if there exists a constant $C > 0$ such that*

$$|\mathscr{F}(x)| \leq C\|x\|$$

for all $x \in V$.

Theorem 6.2. *For every bounded linear functional $\mathscr{F}$ on a Hilbert Space $\mathscr{H}$, there is a uniquely determined element $f \in \mathscr{H}$ such that $\mathscr{F}(x) = \langle x, f \rangle$ for all $x \in \mathscr{H}$.*

Lemma 6.3. *Let $\mathscr{B}$ and $\mathscr{D}$ be Banach spaces and $T : \mathscr{B} \to \mathscr{D}$ be a continuous linear mapping. Suppose that there is a $C > 0$ such that T satisfies*

$$C\|x\| \leq \|T(x)\|$$

for all $x \in \mathscr{B}$. Then, the range of T, $\mathscr{R}(T)$, is a closed set.

Theorem 6.4. *Let $\mathscr{U}$ be a closed subspace of a Hilbert Space $\mathscr{H}$. Then, for every $x \in \mathscr{H}$, we have $x = y + z$ where $y \in \mathscr{U}$ and $z \in \mathscr{U}^{\perp}$.*

6.2 Fredholm Alternative Prerequisite Results

Lemma 6.5. *Let V be a normed vector space and $\mathscr{U}$ a proper closed subspace of V. Then, for any $\theta < 1$, there exists an element $x_\theta \in V$ satisfying $\|x_\theta\| = 1$ and $dist(x_\theta, \mathscr{U}) \geq \theta$.*

7 Appendix C - Young Functions

Young Function	Properties	Sources
$\Psi(t)$	$\Psi(t) = t^2 \log(e-1+t)^q$ $\Psi^{-1}(t) \approx t^{\frac{1}{2}} \log(e-1+t)^{-\frac{q}{2}}$ $\bar{\Psi}(t) \approx t^2 \log(e-1+t)^{-\frac{q}{2}}$ $\bar{\Psi}^{-1}(t) \approx t^{\frac{1}{2}} \log(e-1+t)^{\frac{q}{2}}$	[3], [13]
$\Phi(t)$	$\Phi(t) = t^2 \log(e-1+t)^\gamma$ $\Phi^{-1}(t) \approx t^{\frac{1}{2}} \log(e-1+t)^{-\frac{\gamma}{2}}$ $\bar{\Phi}(t) \approx t^2 \log(e-1+t)^{-\frac{\gamma}{2}}$ $\bar{\Phi}^{-1}(t) \approx t^{\frac{1}{2}} \log(e-1+t)^{\frac{\gamma}{2}}$	[3], [13]
$\Psi_F(t)$	$\Psi_F(t) = e^{t^{\frac{2}{\gamma}}} - 1$ $\Psi_F^{-1}(t) \approx \log(e-1+t)^{\frac{\gamma}{2}}$	[3]
$\Psi_{\mathrm{H}}(t)$	$\Psi_{\mathrm{H}}(t) = e^{t^{\frac{1}{\gamma}}} - 1$ $\Psi_{\mathrm{H}}^{-1}(t) \approx \log(1+t)^\gamma$	[3]
$E(t)$	$E(t) = t \log(e-1+t)^\gamma$ $E^{-1}(t) \approx t \log(e-1+t)^{-\gamma}$ $\bar{E}(t) \approx e^{t^{\frac{1}{\gamma}}} - 1$	[3]
$B(t)$	$B(t) = t \log(e-1+t)^q$ $B^{-1}(t) \approx t \log(e-1+t)^{-q}$	[3]
$\varphi(t)$	$\varphi(t) = e^{t^{\frac{2}{q-\gamma}}}$ $\varphi^{-1}(t) \approx \log(e-1+t)^{\frac{q-\gamma}{2}}$	[3]

Table 1: Section 4.2.2 Young Functions

Remark 7.1. *In Table 1, $q > 0$ and $0 < \gamma \leq \frac{q}{2}$.*

Definition 7.2. *Given two non-decreasing functions a and b, we say b is larger than a and write $a(t) \lesssim b(t)$, if there exists $t_0 \geq 0$ and $c > 0$ such that for all $t \geq t_0$, then $a(t) \leq cb(t)$. If $a(t) \lesssim b(t)$ and $b(t) \lesssim a(t)$, we write $a(t) \approx b(t)$.*

Definition 7.3. *Given two non-decreasing functions a and b, we say b dominates a and write $a(t) \preceq b(t)$, if there exists $t_0 \geq 0$ and $c > 0$ such that for all $t \geq t_0$, $a(t) \leq b(ct)$. If $a(t) \preceq b(t)$ and $b(t) \preceq a(t)$, we write $a(t) \sim b(t)$.*

Lemma 7.4. *Let f be a measurable function. Then, $\|f\|_{L^\Phi(\Omega;w)} \leq 2^q \|f\|_{L^\Psi(\Omega;w)}$ for $q > 0$.*

Proof. Since $e + t \leq (e - 1 + t)^2$ for all $t \geq 0$ and $0 < \gamma \leq \frac{q}{2}$, we have that

$$\begin{aligned}
\Phi(t) &= t^2(\log(e - 1 + t))^\gamma \\
&\leq t^2(\log(e + t))^\gamma \\
&\leq t^2(\log(e + t))^q \\
&\leq t^2(\log((e - 1 + t)^2))^q \\
&= 2^q t^2(\log(e - 1 + t))^q \\
&= 2^q \Psi(t).
\end{aligned}$$

Since $2^q \geq 1$ for all $q > 0$, Proposition 2.23 concludes the proof.

□

Lemma 7.5. *Let $\Psi^{-1}(t), \Psi_F^{-1}(t)$ and $\bar{\Psi}^{-1}(t)$ be defined as in Table 1. Then,*

$$\Psi^{-1}(t)\Psi_F^{-1}(t) \leq \bar{\Psi}^{-1}(ct) \tag{7.1}$$

for all $t \geq 0$ where $c = \dfrac{2^q}{\log(e-1)^q}$ and $q > 0$.

Proof. Notice,

$$\log(e - 1 + t)^{\frac{\gamma}{2}} \leq \log(e + t)^{\frac{\gamma}{2}} \leq \log(e + t)^{\frac{q}{2}} \leq 2^{\frac{q}{2}} \log(e - 1 + t)^{\frac{q}{2}}.$$

Consequently,

$$\begin{aligned}
\frac{\log(e - 1 + t)^{\frac{\gamma}{2}}}{\log(e - 1 + t)^{\frac{q}{2}} \log(e - 1 + ct)^{\frac{q}{2}}} &\leq 2^{\frac{q}{2}} \frac{\log(e - 1 + t)^{\frac{q}{2}}}{\log(e - 1 + t)^{\frac{q}{2}} \log(e - 1 + ct)^{\frac{q}{2}}} \\
&= \frac{2^q}{\log(e - 1 + ct)^{\frac{q}{2}}}.
\end{aligned}$$

We want to find $c > 0$ such that

$$\frac{2^q}{\log(e - 1 + ct)^{\frac{q}{2}}} \leq c^{\frac{1}{2}}.$$

Equivalently,

$$2^q \leq c^{\frac{1}{2}} \log(e - 1 + ct)^{\frac{q}{2}}. \tag{7.2}$$

Since $\log(e - 1 + ct)^{\frac{q}{2}}$ is increasing as c increases, it suffices to choose $c > 0$ that satisfies (7.2) at the minimum value. That is, $t = 0$. This restricts our choice of c to satisfy

$$c \geq \frac{2^q}{\log(e-1)^q}.$$

Choosing $c = \dfrac{2^q}{\log(e-1)^q}$ establishes (7.1) for all $t \geq 0$.

□

Lemma 7.6. *Let $B^{-1}(t)$ and $E^{-1}(t)$ be defined as in Table 1. Then,*

$$t^{\frac{1}{2}}(B^{-1}(t))^{\frac{1}{2}} \leq E^{-1}(ct) \tag{7.3}$$

holds for all $t \geq 1$, where $c = 1$ and $q > 0$.

Proof. We want to find a $c > 0$ such that

$$\frac{\log(e-1+ct)^\gamma}{\log(e-1+t)^{\frac{q}{2}}} \leq c.$$

Since $0 < \frac{\gamma}{2} \leq q$, setting $c = 1$ ensures

$$\frac{\log(e-1+t)^\gamma}{\log(e-1+t)^{\frac{q}{2}}} \leq 1$$

for all $t \geq 1$.

□

References

[1] Scott Rodney. Existence of Weak Solutions of Linear Subelliptic Dirichlet Problems with Rough Coefficients. *Canadian Journal of Mathematics*, 64(6):1395–1414, 2012.

[2] Amos Ron and Zuowei Shen. Frames and Stable Bases for Shift-Invariant Subspaces of $[L^2(\mathbb{R})]^d$. *Canadian Journal of Mathematics*, 47(5):1051–1094, 1995.

[3] David Cruz-Uribe and Kabe Moen. The Theory of Weighted Norm Inequalities. Preprint available upon request from David Cruz-Uribe, University of Alabama, AL USA.

[4] H.L. Royden. *Real Analysis 3rd Ed.* Prentice-Hall Of India Pvt. Limited, 1988.

[5] Emily Rosta. Relating the p-Poincaré Inequality to L^p Hypo-Ellipticity for the Homogeneous Neumann Problem for the p-Laplacian. https://cbu.scholaris.ca/items/c5fd7c18-444e-4402-83e4-45cc47f6d8e7, 2016.

[6] Eugene B Fabes, Carlos E Kenig, and Raul P Serapioni. The Local Regularity of Solutions of Degenerate Elliptic Equations. *Communications in Statistics-Theory and Methods*, 7(1):77–116, 1982.

[7] Dario D. Monticelli and Scott Rodney. An Improved Compact Embedding Theorem for Degenerate Sobolev Spaces. *Matematiche (Catania)*, 75(1):259–275, 2020.

[8] David Gilbarg, Neil S Trudinger, David Gilbarg, and NS Trudinger. *Elliptic Partial Differential Equations of Second Order*, volume 224. Springer, 1977.

[9] Scott Rodney. A Degenerate Sobolev Inequality for a Large Open Set in a Homogenerous Space. *Transactions of the American Mathematical Society*, 362(2):673–685, 2010.

[10] Scott Rodney. Existence of Weak Solutions to Subelliptic Partial Differential Equations in Divergence form and the Necessity of the Sobolev

and Poincaré Inequalities. ProQuest LLC, Ann Arbor, MI, 2007. Thesis (Ph.D.)–McMaster University (Canada).

[11] Sullivan MacDonald. Boundedness of Weak Solutions to Degenerate Elliptic Equations. https://cbu.scholaris.ca/items/719d3a14-59b8-4554-b254-f7e0a960a143, 2021.

[12] Lawrence C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[13] David Cruz-Uribe and Scott Rodney. Bounded Weak Solutions to Elliptic PDE with Data in Orlicz Spaces. *Journal of Differential Equations*, 297:409–432, 2021.